Cardinality

Outline for Today

- ***Bijections***
  - A key and important class of functions.
- ***Cardinality, Formally***
  - What does it mean for two sets to have the same size?
- ***Cantor's Theorem, Formally***
  - Proving, indeed, that infinity is not infinity is not infinity.

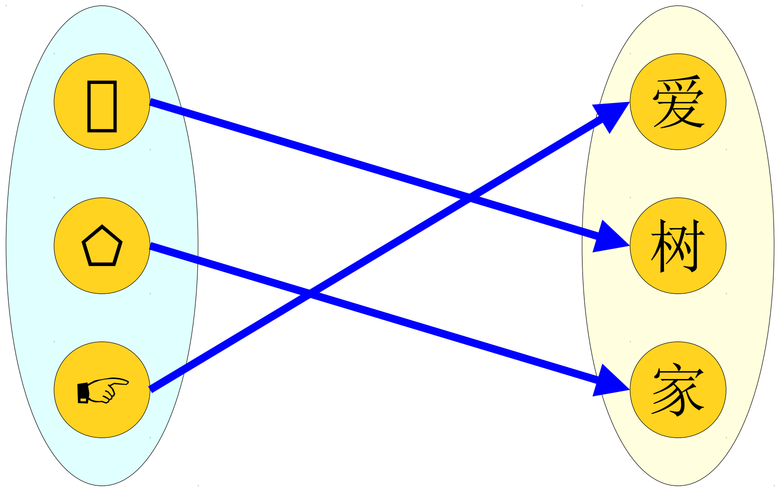
Bijections

Injections and Surjections

- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- What about functions that associate ***exactly one*** element of the domain with each element of the codomain?

Bijections

- A ***bijection*** is a function that is both injective and surjective.
- Intuitively, if $f : A \to B$ is a bijection, then f represents a way of pairing off elements of A and elements of B.



Bijections

- Which of the following are bijections?
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$.
  - $f : \mathbb{Z} \to \mathbb{R}$ defined as $f(x) = x$.
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 2x + 1$.
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = 2x + 1$.

A ***bijection*** is a function that is both injective and surjective.

Bijections

- Which of the following are bijections?
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{R}$ defined as $f(x) = x$. *Nope!*
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 2x + 1$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = 2x + 1$. *Nope!*

A ***bijection*** is a function that is both injective and surjective.

Cardinality Revisited

Cardinality

- Recall *(from our first lecture!)* that the ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality by $|S|$.
- For finite sets, cardinalities are natural numbers:
  - $|\{1, 2, 3\}| = 3$
  - $|\{100, 200\}| = 2$
- For infinite sets, we introduced ***infinite cardinals*** to denote the size of sets:

$$|\mathbb{N}| = \aleph_0$$

Defining Cardinality

- It is difficult to give a rigorous definition of what cardinalities actually are.
  - What is 4? What is $\aleph_0$?
  - (Take Math 161 for an answer!)
- Instead, we'll define cardinality as a *relationship* between two sets rather than an absolute quantity.
- ***Intuition:*** Two sets have the same cardinality if there's a way to pair off their elements.

Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

$|S| = |T|$ if there exists a *bijection* $f : S \to T$

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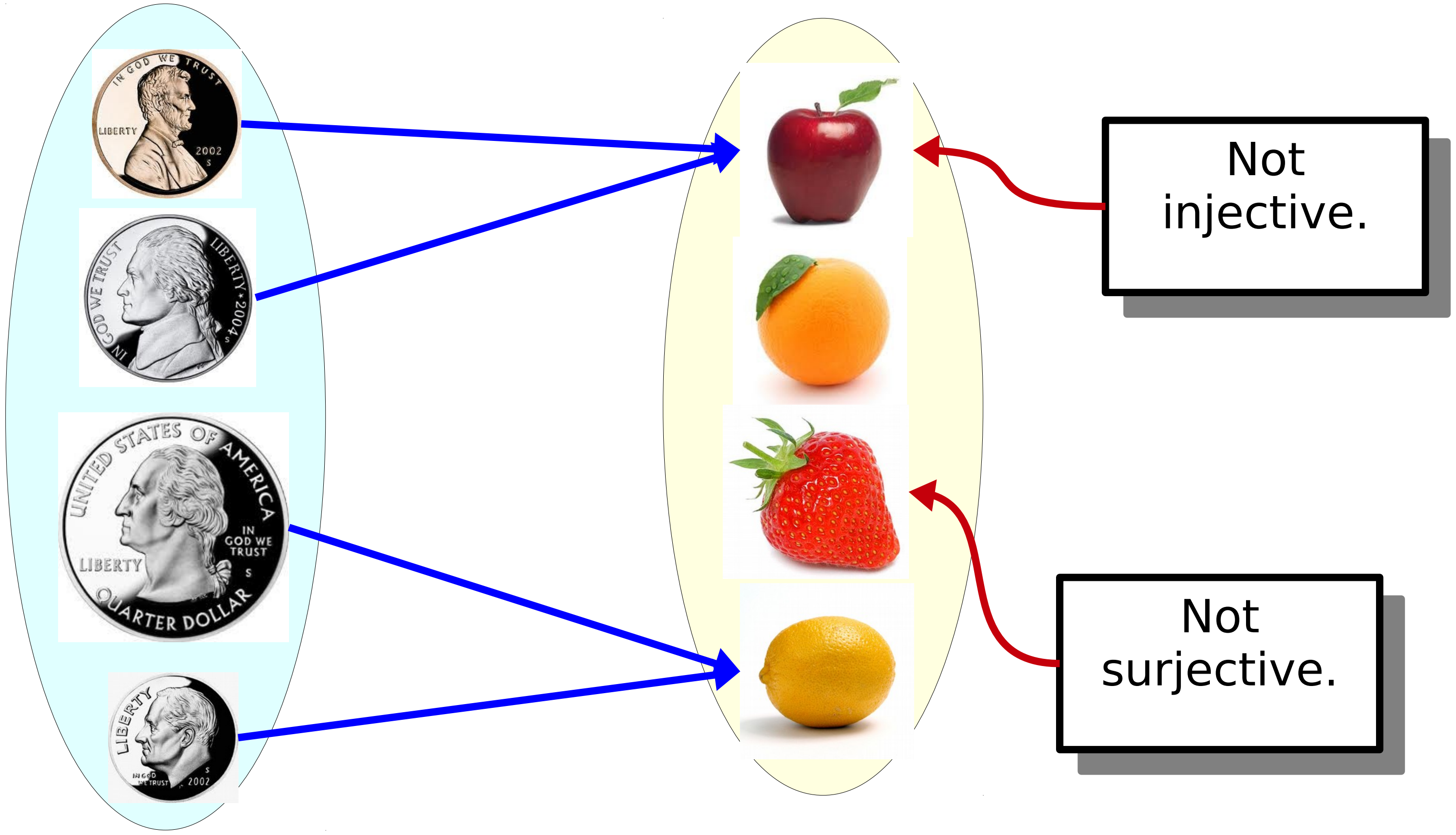

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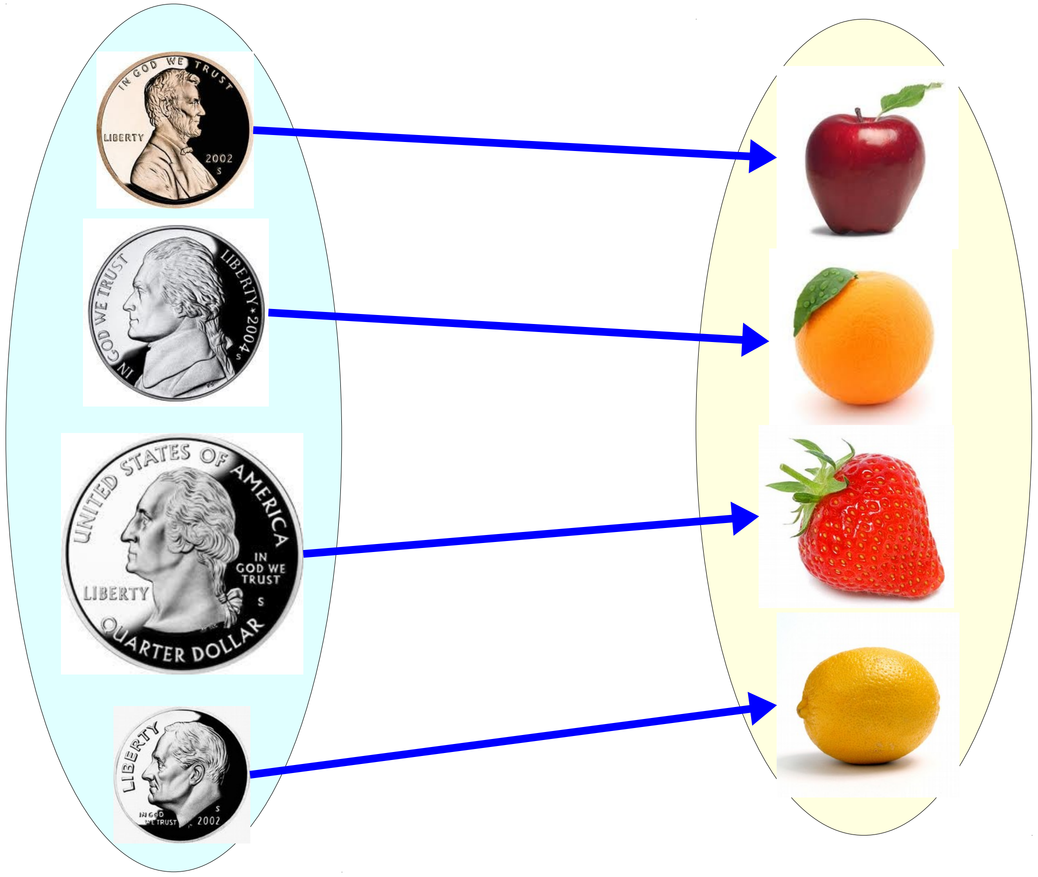


Yay, bijective!

Fun with Cardinality

Terminology Refresher

- Let a and b be real numbers where $a \leq b$.
- The notation $[a, b]$ denotes the set of all real numbers between a and b, inclusive.

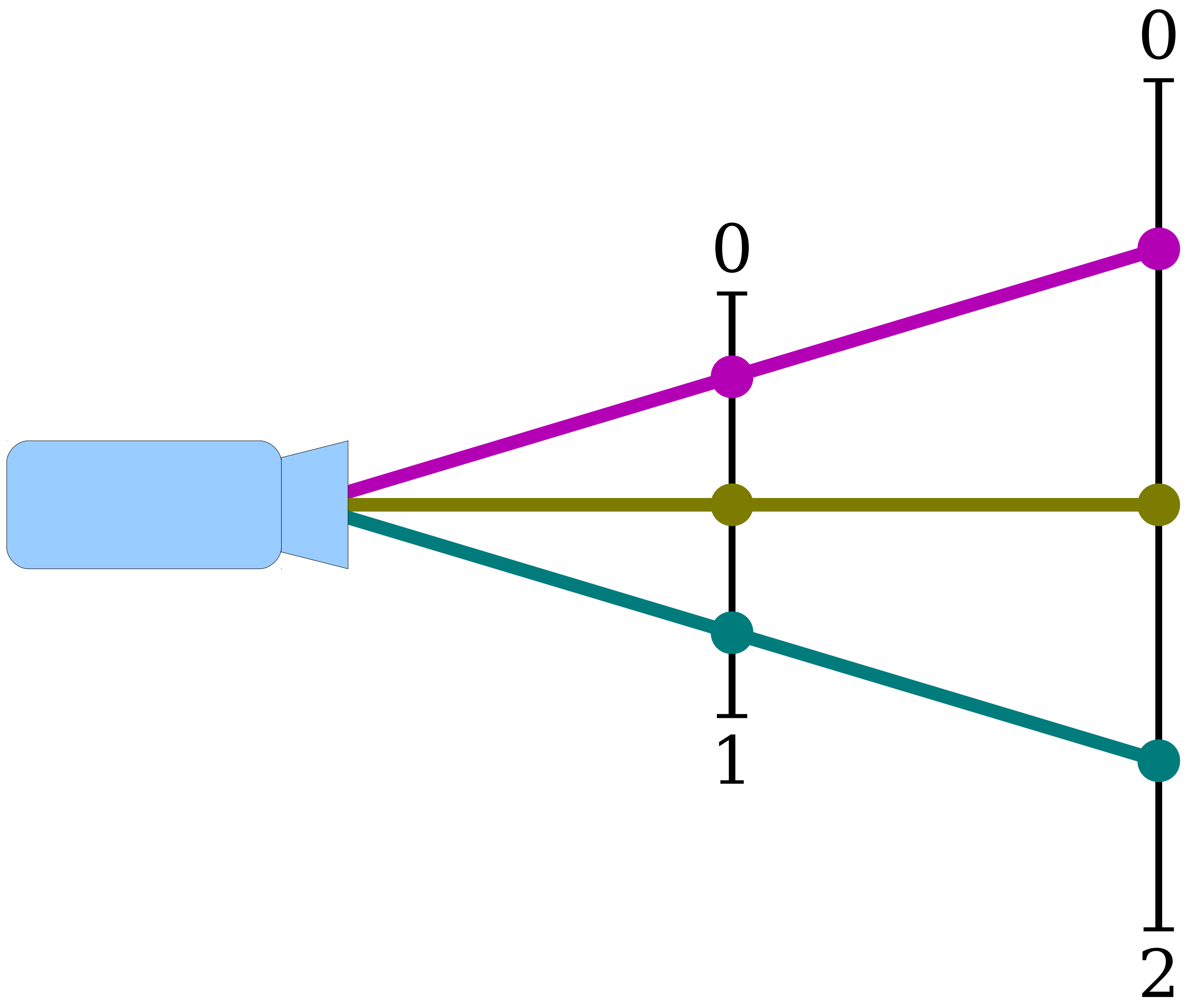
$$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

- The notation (a, b) denotes the set of all real numbers between a and b, exclusive.

$$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$$

Consider the sets $[0, 1]$ and $[0, 2]$.

How do their cardinalities compare?



$f : [0, 1] \to [0, 2]$

$f(x) = 2x$

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Finally, we will show tha … ny
$y \in [0, 2]$. We'll show that … .

When defining something we claim is a function, the convention is to prove that it obeys the domain/codomain rules. For whatever reason, there isn't a convention of showing that it's deterministic. Ah, tradition. 😃

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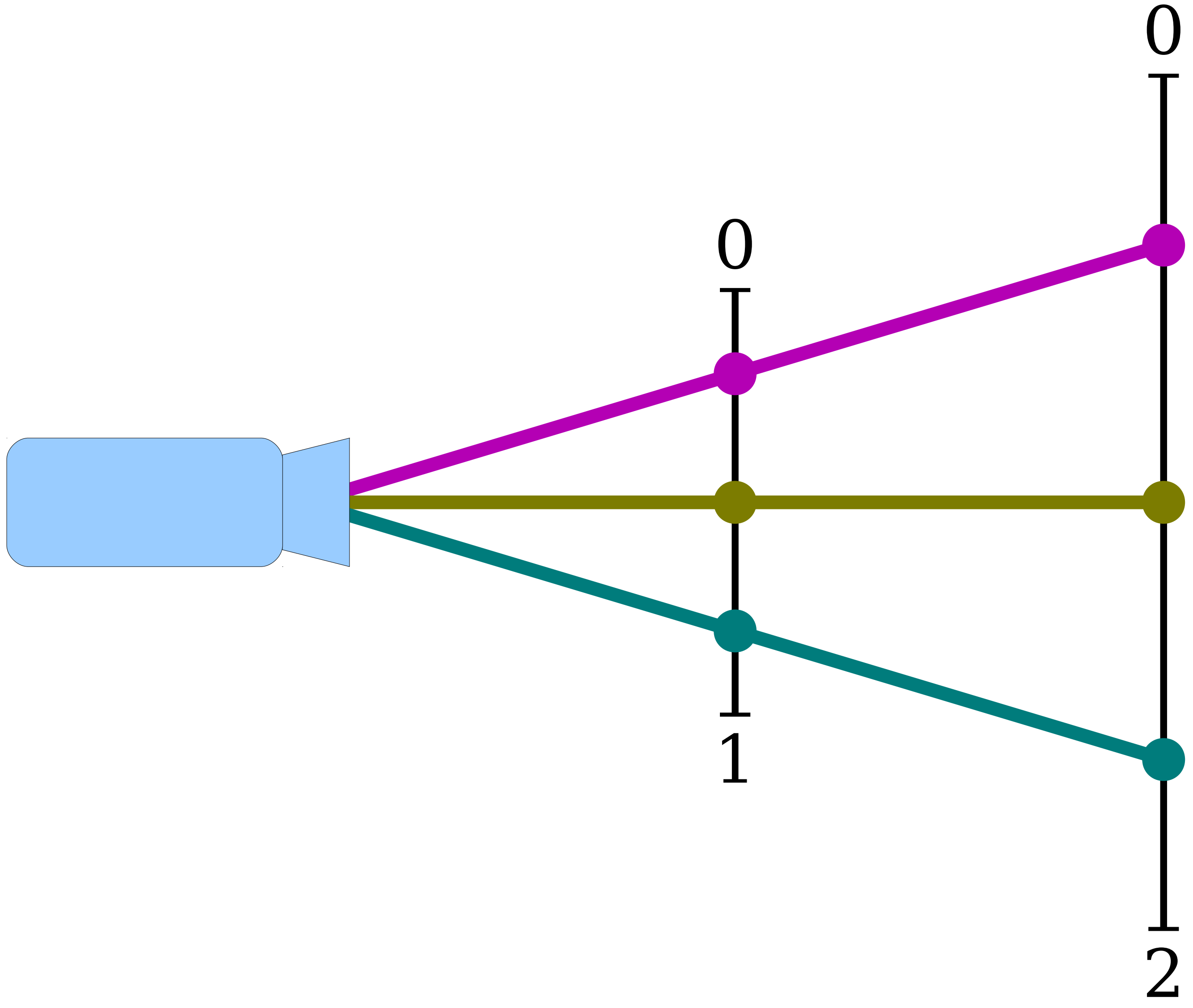

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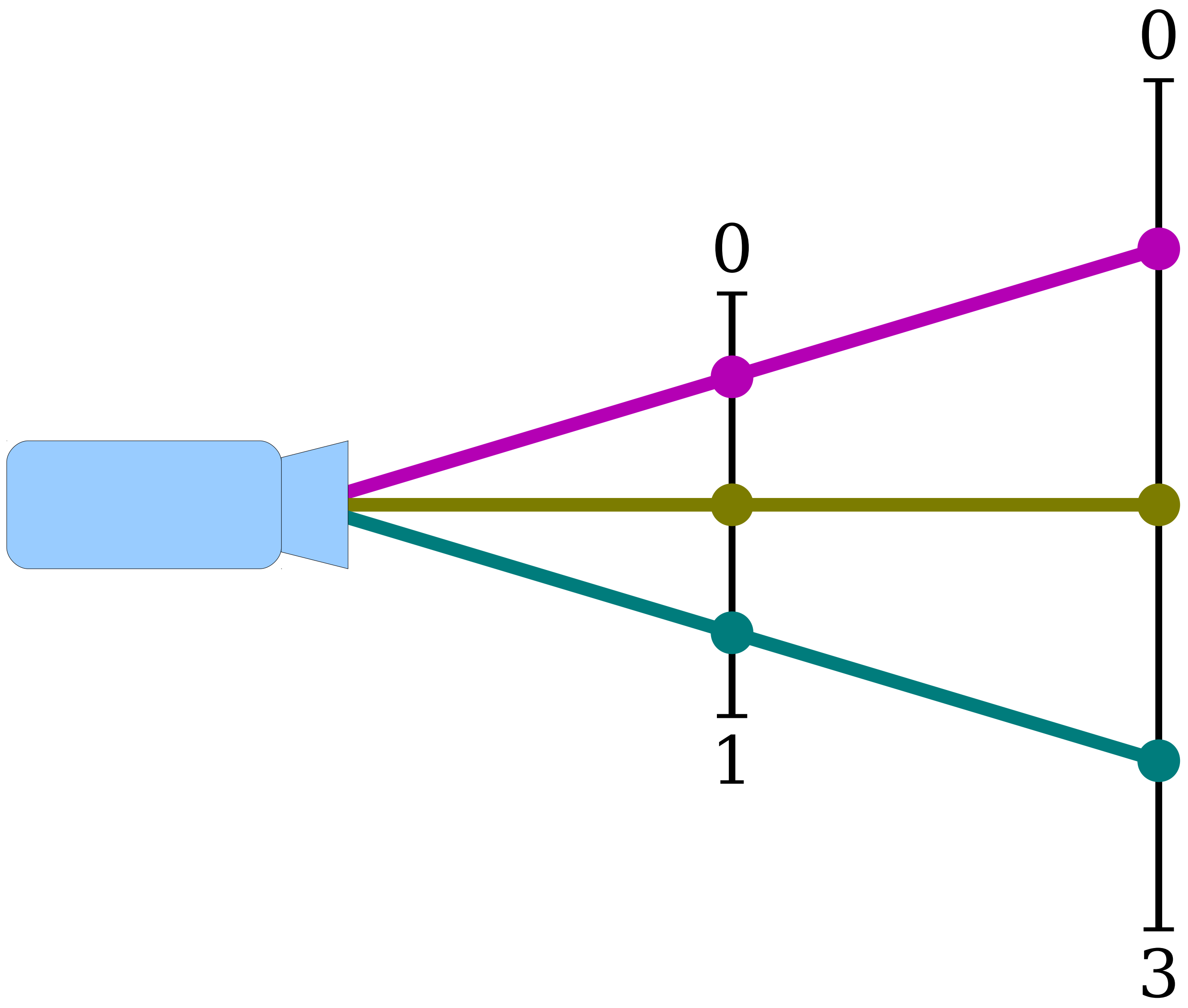

$$f : [0, 1] \to [0, 3]$$
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Theorem: $|[0, 1]| = |[0, 3]|$

Proof: Consider the function $f : [0, 1] \to [0, 3]$ defined as $f(x) = 3x$. We will prove that f is a bijection.

First, we will show that f is a well-defined function. Choose any $x \in [0, 1]$. This means that $0 \leq x \leq 1$, so we know that $0 \leq 3x \leq 3$. Consequently, we see that $0 \leq f(x) \leq 3$, so $f(x) \in [0, 3]$.

Next, we'll show that f is injective. Pick any $x_1, x_2 \in [0, 1]$ where $f(x_1) = f(x_2)$. We will show that $x_1 = x_2$. To see this, notice that since $f(x_1) = f(x_2)$, we see that $3x_1 = 3x_2$, which in turn tells us that $x_1 = x_2$, as required.

Finally, we will show that f is surjective. To do so, consider any $y \in [0, 3]$. We'll show that there is some $x \in [0, 1]$ where $f(x) = y$.

Let $x = y/3$. Since $y \in [0, 3]$, we know $0 \leq y \leq 3$, and therefore that $0 \leq y/3 \leq 1$. We picked $x = y/3$, so we know that $0 \leq x \leq 1$, which in turn means $x \in [0, 1]$. Moreover, notice that

$$f(x) = 3x = 3(y/3) = y,$$

so $f(x) = y$, as required. ■

Theorem: $|[0, 1]| = |[0, 2]|$

Proof: Consider the function $f : [0, 1] \to [0, 2]$ defined as $f(x) = 2x$. We will prove that f is a bijection.

First, we will show that f is a well-defined function. Choose any $x \in [0, 1]$. This means that $0 \leq x \leq 1$, so we know that $0 \leq 2x \leq 2$. Consequently, we see that $0 \leq f(x) \leq 2$, so $f(x) \in [0, 2]$.

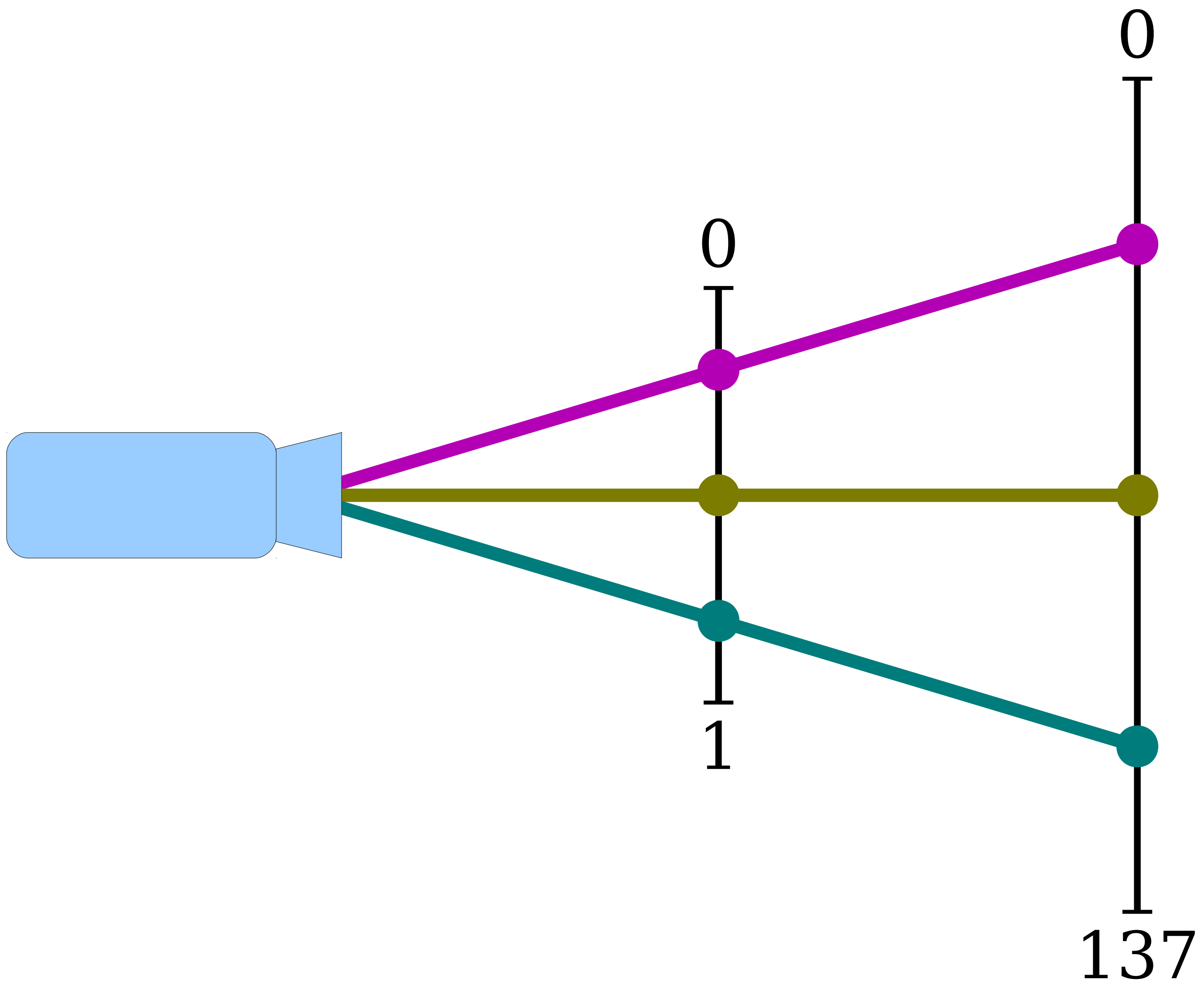
Next, we'll show that f is injective. Pick any $x_1, x_2 \in [0, 1]$ where $f(x_1) = f(x_2)$. We will show that $x_1 = x_2$. To see this, notice that since $f(x_1) = f(x_2)$, we see that $2x_1 = 2x_2$, which in turn tells us that $x_1 = x_2$, as required.

Finally, we will show that f is surjective. To do so, consider any $y \in [0, 2]$. We'll show that there is some $x \in [0, 1]$ where $f(x) = y$.

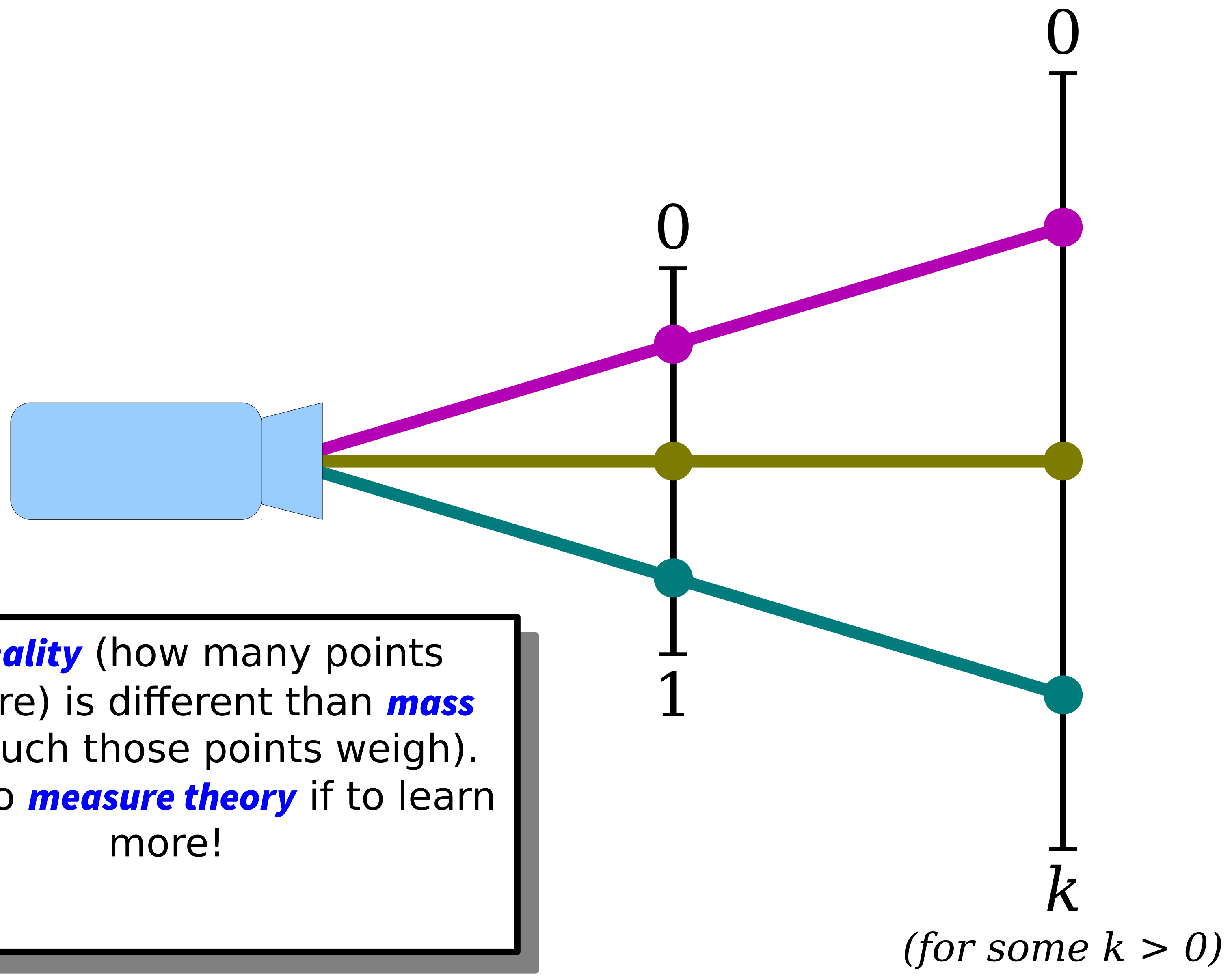
Let $x = {}^{y}/_{2}$. Since $y \in [0, 2]$, we know $0 \leq y \leq 2$, and therefore that $0 \leq {}^{y}/_{2} \leq 1$. We picked $x = {}^{y}/_{2}$, so we know that $0 \leq x \leq 1$, which in turn means $x \in [0, 1]$. Moreover, notice that

$$f(x) = 2x = 2({}^{y}/_{2}) = y,$$

so $f(x) = y$, as required. ■



$$f : [0, 1] \to [0, 137]$$
$$f(x) = 137x$$



Cardinality (how many points there are) is different than ***mass*** (how much those points weigh). Look into ***measure theory*** if to learn more!

$$f : [0, 1] \to [0, k]$$
$$f(x) = kx$$

Some Properties of Cardinality

Theorem: For any set A, we have $|A| = |A|$.

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Proof: Consider any set A, and let $f : A \to A$ be the function defined as $f(x) = x$. We will prove that f is a bijection.

First, we'll show that f is a well-defined function. To see this, note that for any $x \in A$, we have $f(x) = x \in A$, as needed.

Next, we'll show that f is injective. Pick any $x_1, x_2 \in A$ where $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. Since $f(x_1) = f(x_2)$, we see by definition of f that $x_1 = x_2$, as required.

Finally, we'll show that f is surjective. Consider any $y \in A$. We will prove that there is some $x \in A$ where $f(x) = y$. Pick $x = y$. Then $x \in A$ (since $y \in A$) and $f(x) = x = y$, as required. ■

Theorem: If A, B, and C are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

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Proof: Consider any sets A, B, and C where $|A| = |B|$ and $|B| = |C|$. We need to prove that $|A| = |C|$. To do so, we need to show that there is a bijection from A to C.

Since $|A| = |B|$, we know that there is a some bijection $f : A \to B$. Similarly, since $|B| = |C|$ we know that there is at least one bijection $g : B \to C$.

Consider the function $g \circ f : A \to C$. Since g and f are bijections and the composition of two bijections is a bijection, we see that $g \circ f$ is a bijection from A to C. Thus $|A| = |C|$, as required. ■

Great exercise: Prove that if A and B are sets where $|A| = |B|$, then $|B| = |A|$.

Cantor’s Theorem Revisited

Cantor's Theorem

- In our very first lecture, we sketched out a proof of ***Cantor's theorem***, which says that

If S is a set, then $|S| < |\wp(S)|$.

- That proof was visual and pretty hand-wavy. Let's see if we can go back and formalize it!

Where We're Going

- Today, we're going to formally prove the following result:

If S is a set, then $|S| \neq |\wp(S)|$.

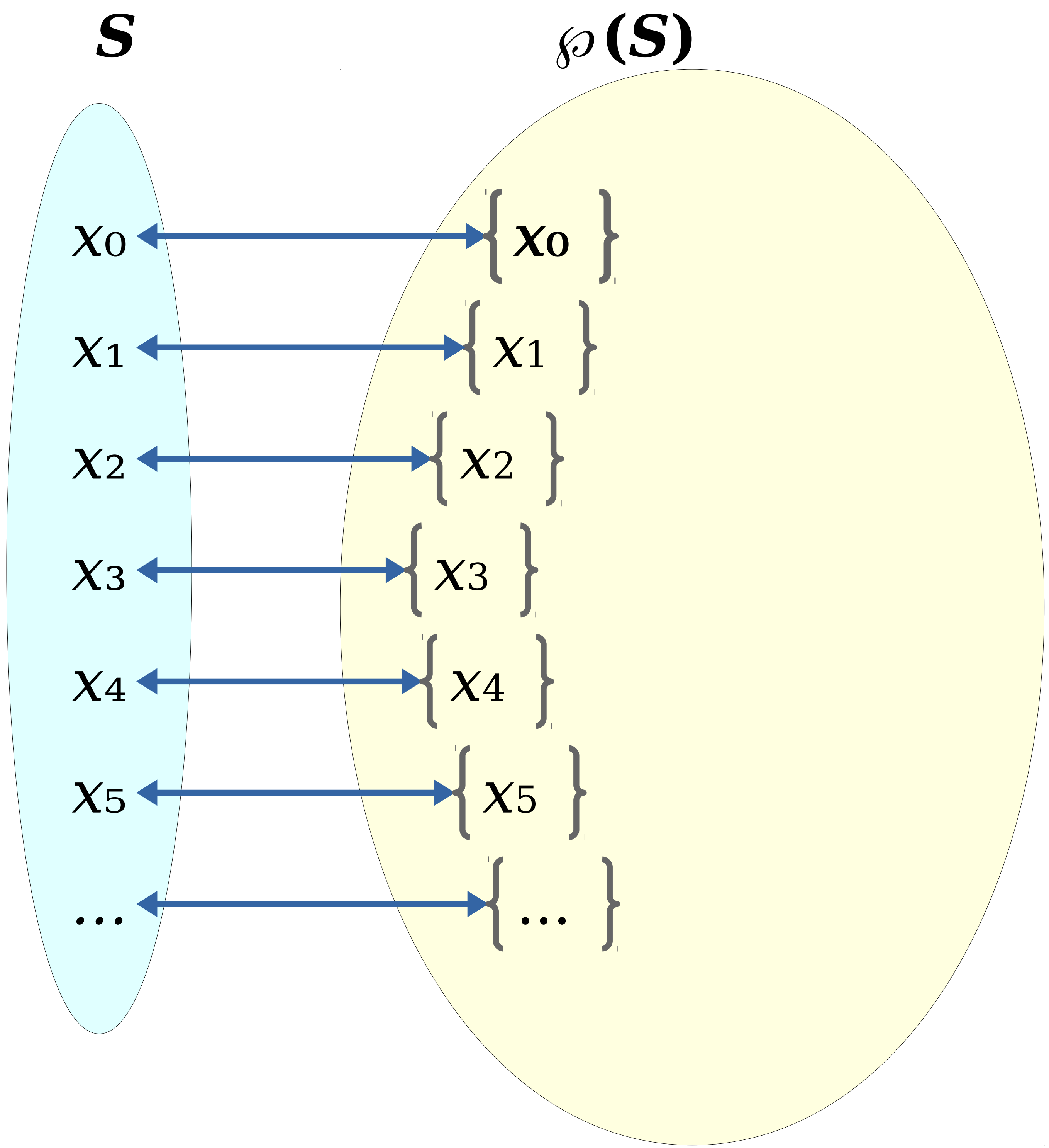
- The goal for today will be to see how to start with our picture and turn it into something rigorous.
- Further exploration: On the problem set, you'll explore the proof in more depth and see some other applications.
- Further reading: Guide to Cantor's Theorem, on the course website

Bijection and Cardinality

- If we think this is true for some set S:

$$|S| \neq |\wp(S)|$$

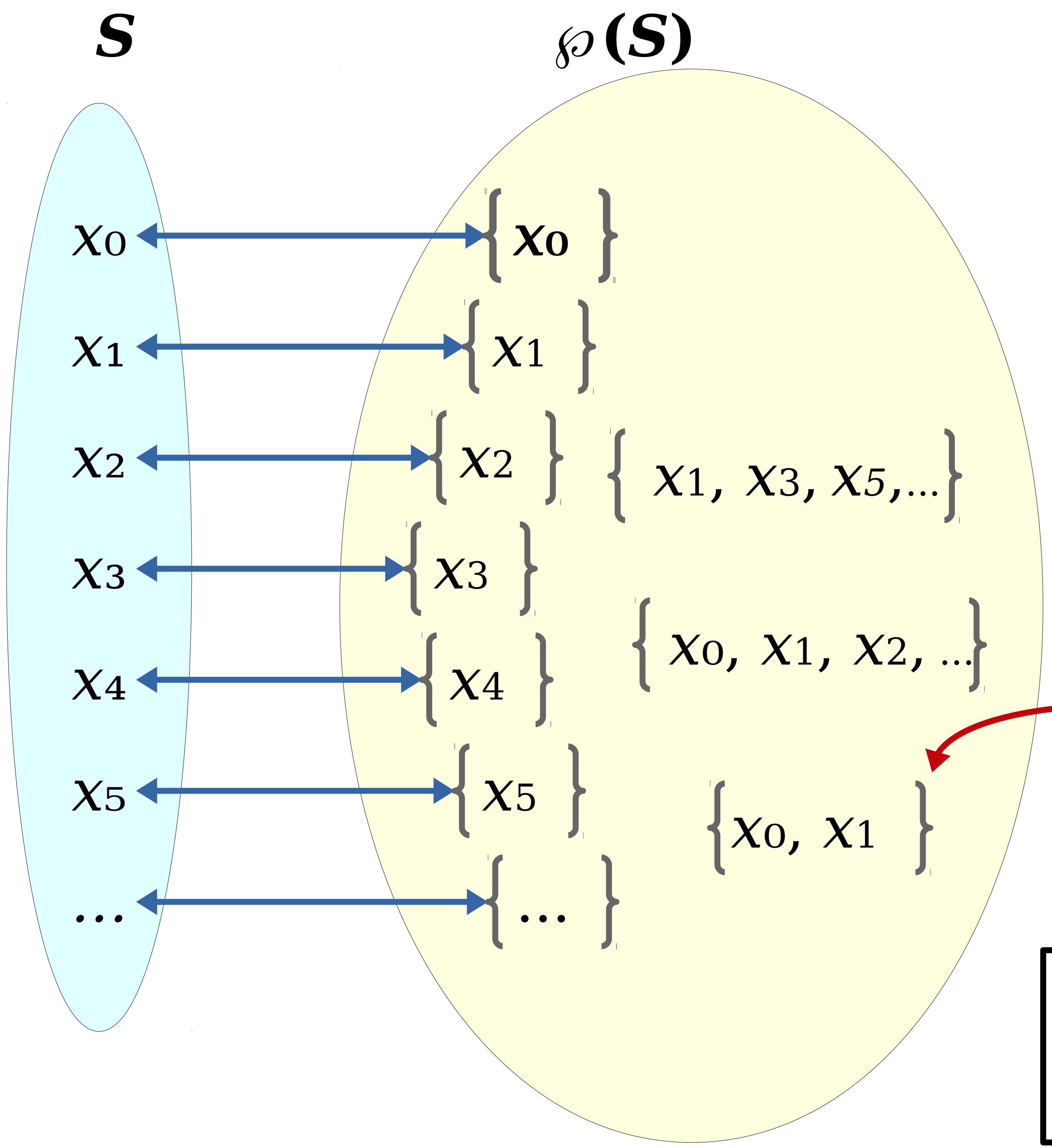
- Then we're saying we don't believe that there exists a bijection between $\boldsymbol{S}$ and $\wp(\boldsymbol{S})$.
- Let's explore one example function from $\boldsymbol{S}$ to $\wp(\boldsymbol{S})$.
  - (remember: we aren't expecting that this can be a bijection)



This is a drawing of a function $f : S \to \wp(S)$*, where* $f(x) = \{x\}$.

In other words, we map each element onto a singleton set containing just itself.

This function is injective.



This is a drawing of a function $f : S \to \wp(S)$, where $f(x) = \{x\}$.
In other words, we map each element onto a singleton set containing just itself.

Not surjective.

(As we expected, this *f* is not bijective.)

Bijection and Cardinality

- Ok we found one function $f : S \to \wp(S)$, where $f(x) = \{x\}$, and showed that this function is not bijective.
- **Question:** Have we proved this?

$$|S| \neq |\wp(S)|$$

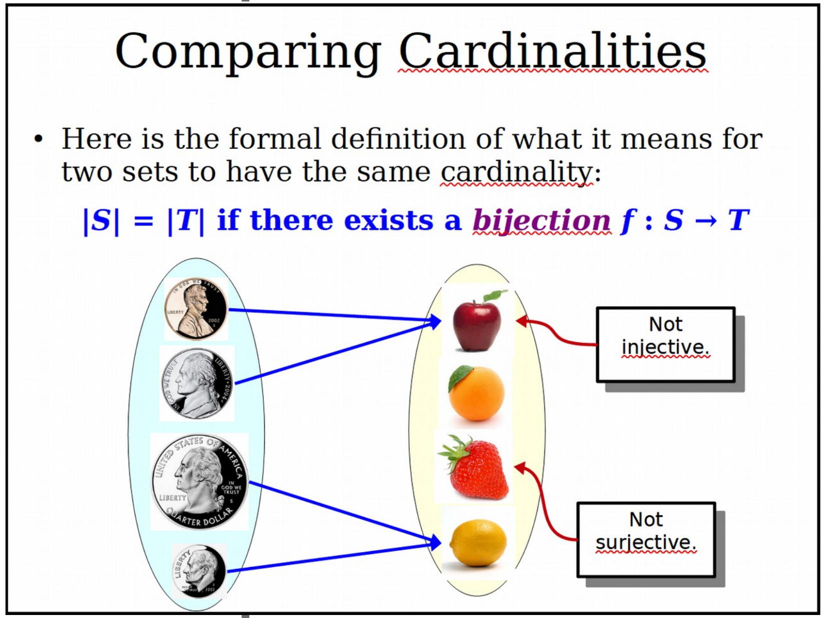
- Why or why not?

Bijection and Cardinality

- Ok we found one function $f : S \to \wp(S)$, where $f(x) = \{x\}$, and showed that this function is not bijective.
- **Question:** Have we proved this?

$$|S| \neq |\wp(S)|$$

- **Answer:** No, because there could be some other function that is bijective.
- Remember our coins/fruit slide from earlier!



The Roadmap

- We're going to prove this statement:

$$\text{If } S \text{ is a set, then } |S| \neq |\wp(S)|.$$

- Here's how this will work:
  - Pick an arbitrary set S.
  - Pick an **arbitrary** function $f : S \to \wp(S)$.
  - Show that f is not surjective using a diagonal argument.
  - Conclude that there are no bijections from S to $\wp(S)$.
  - Conclude that $|S| \neq |\wp(S)|$.

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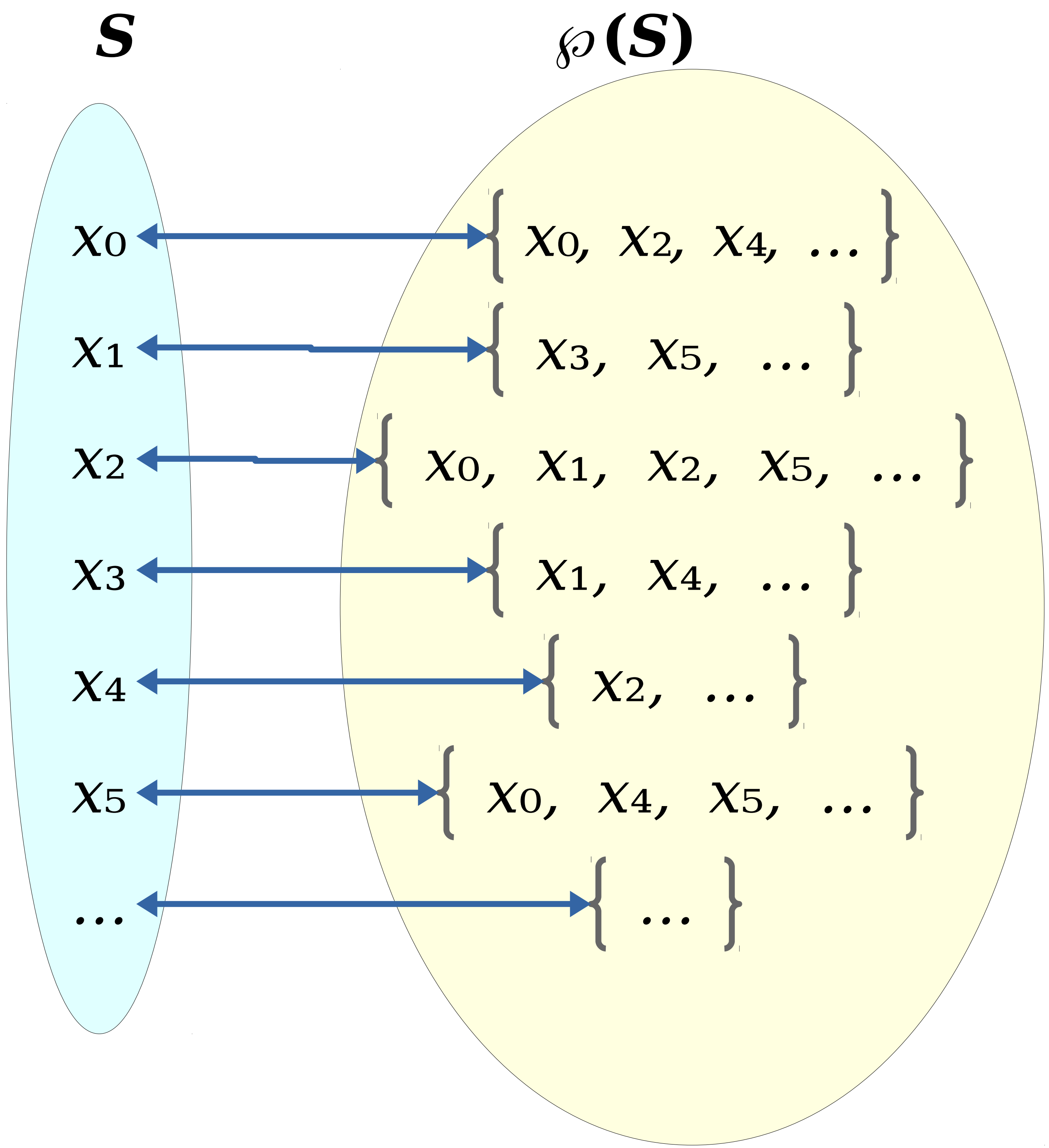
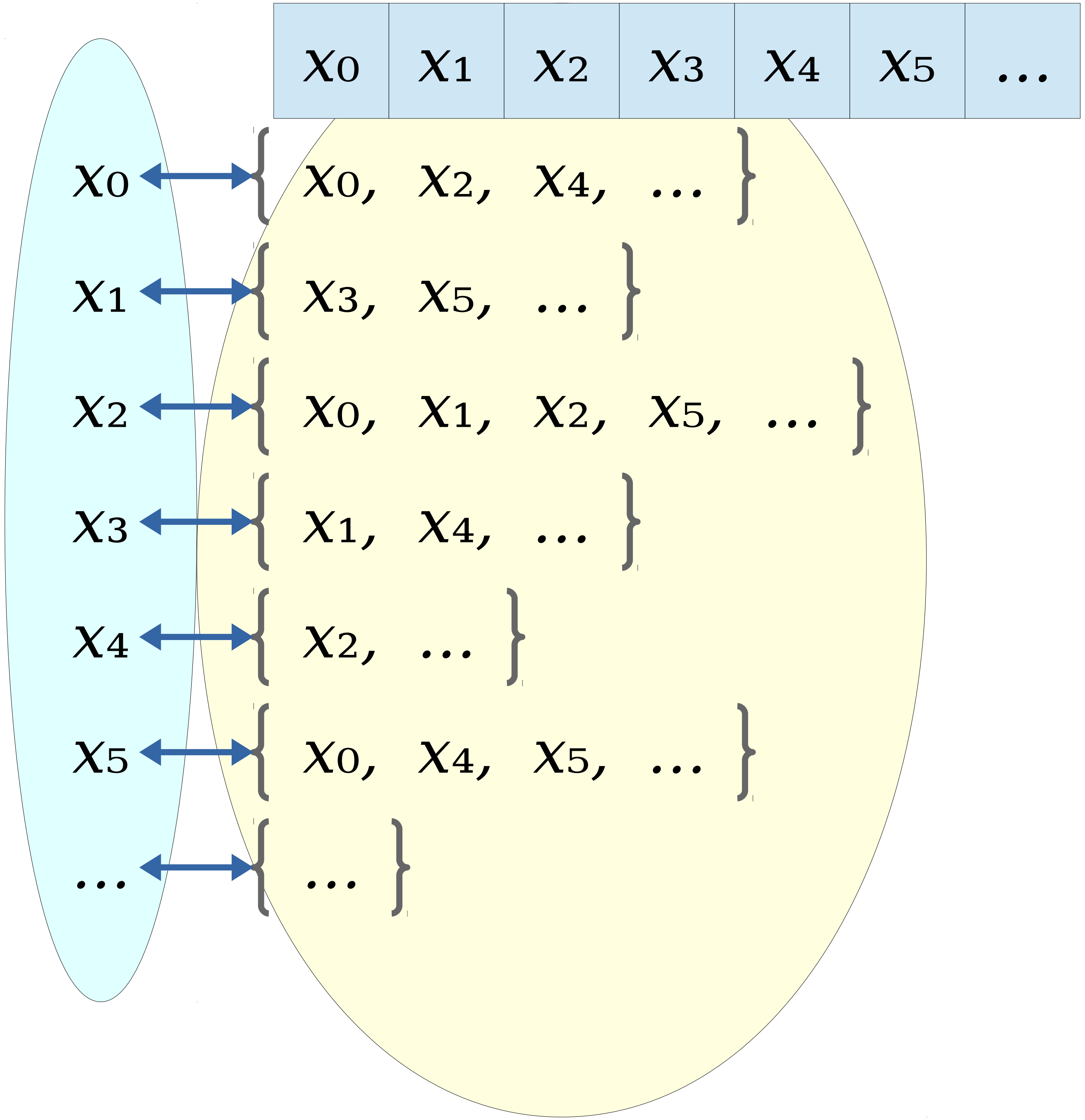


*For this proof, we pick an **arbitrary** function $f : S \to \wp(S)$. We don't know what f looks like, so this drawing just has some "random" values as examples of what the f might look like.*



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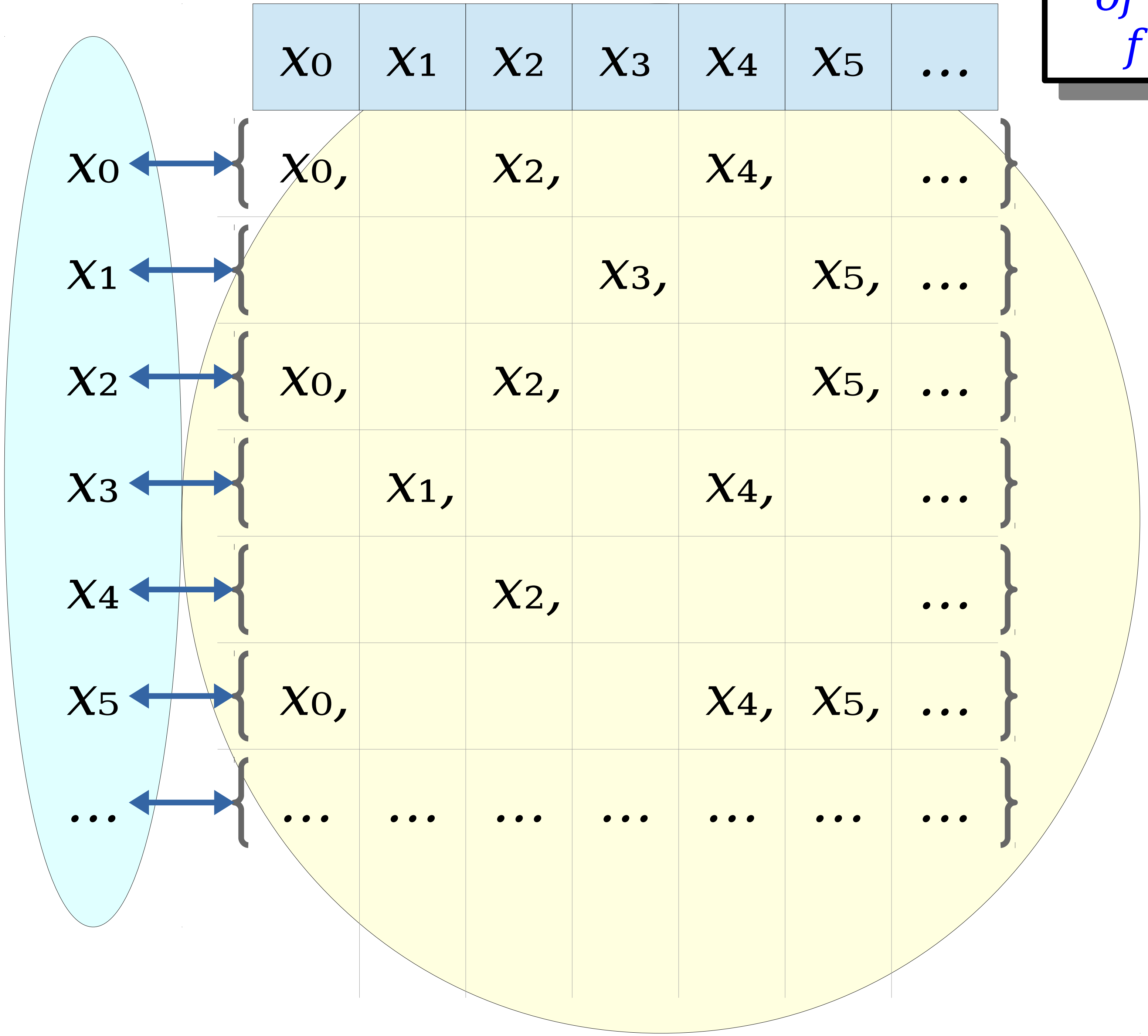

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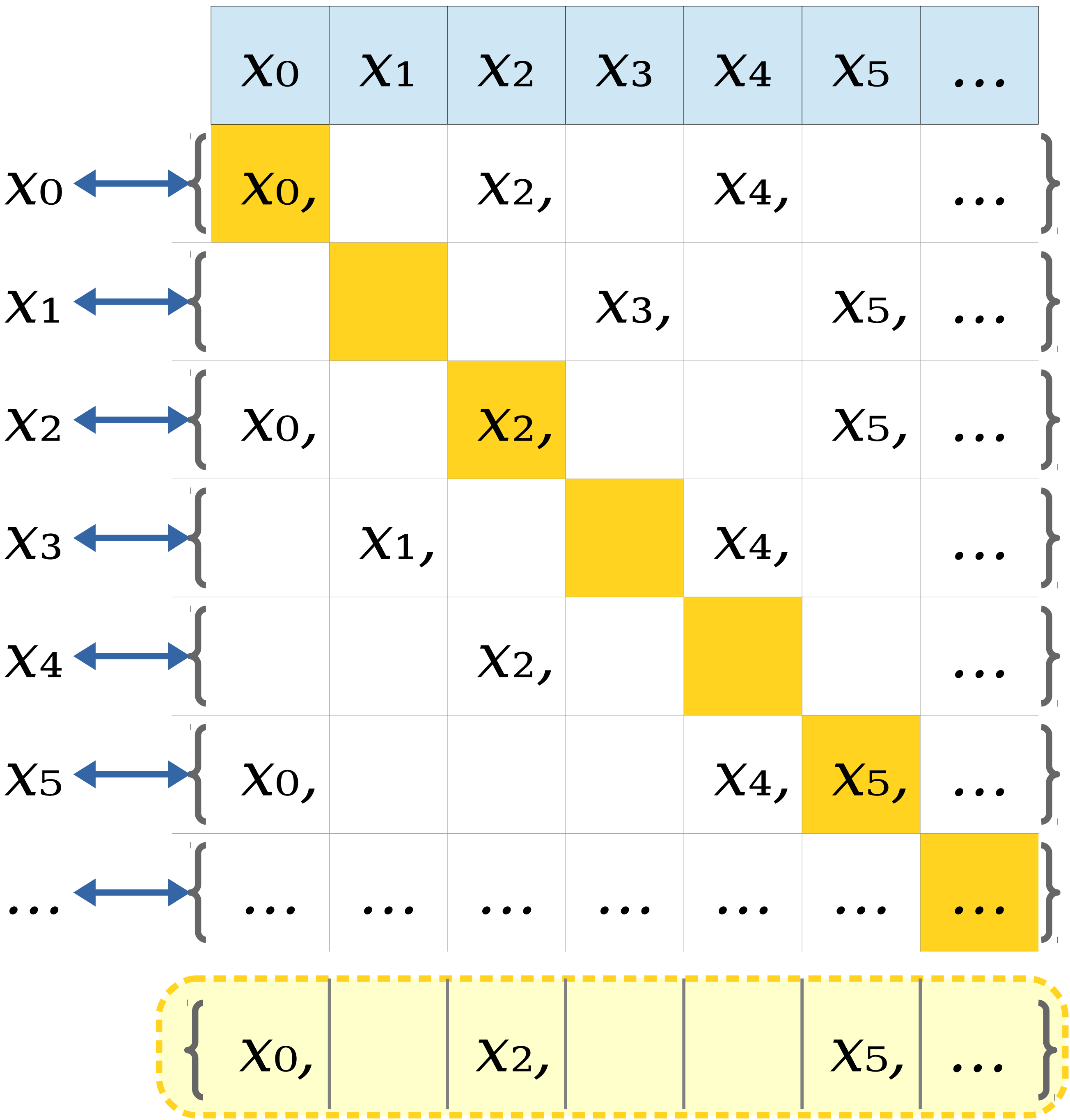

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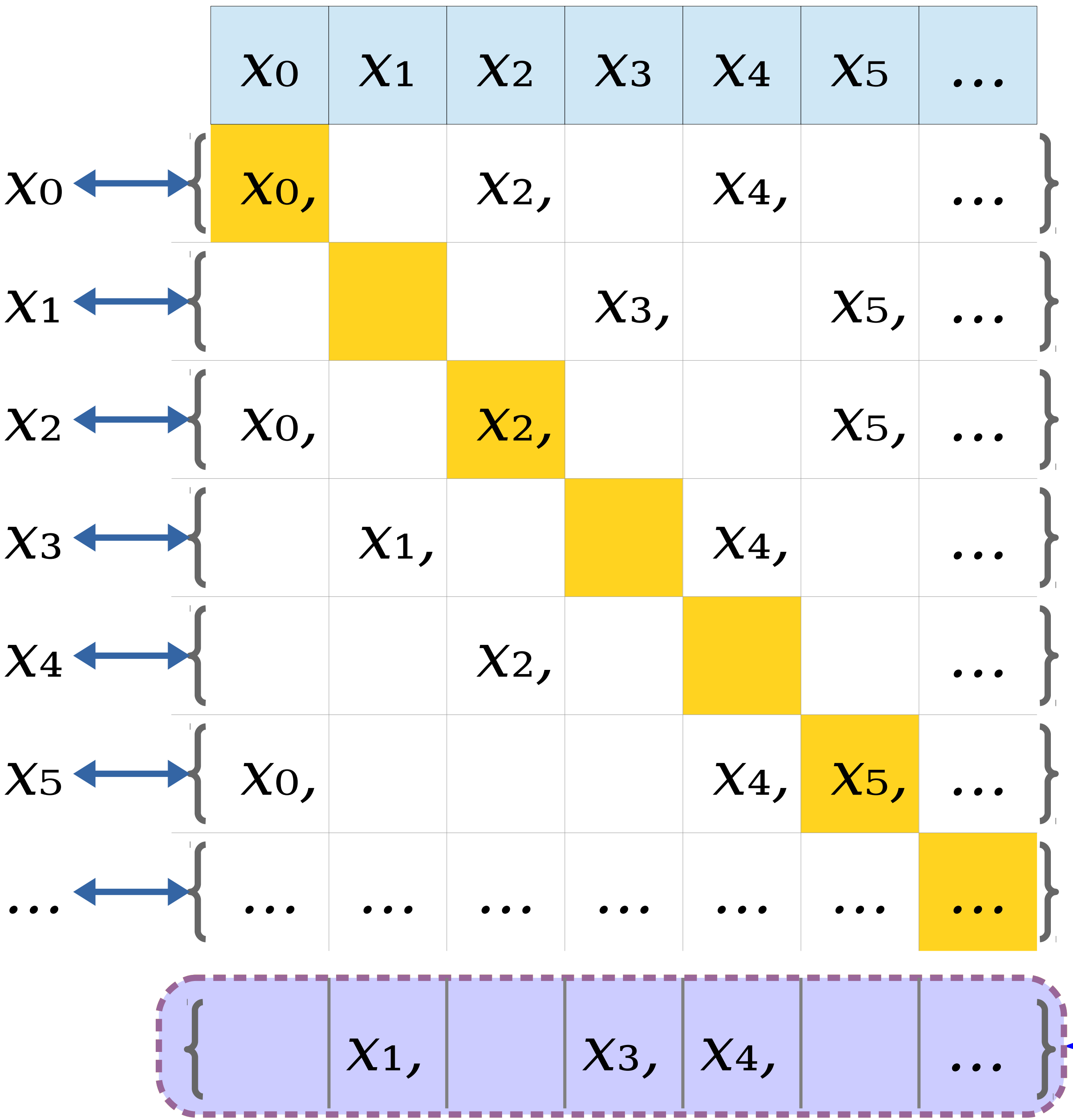
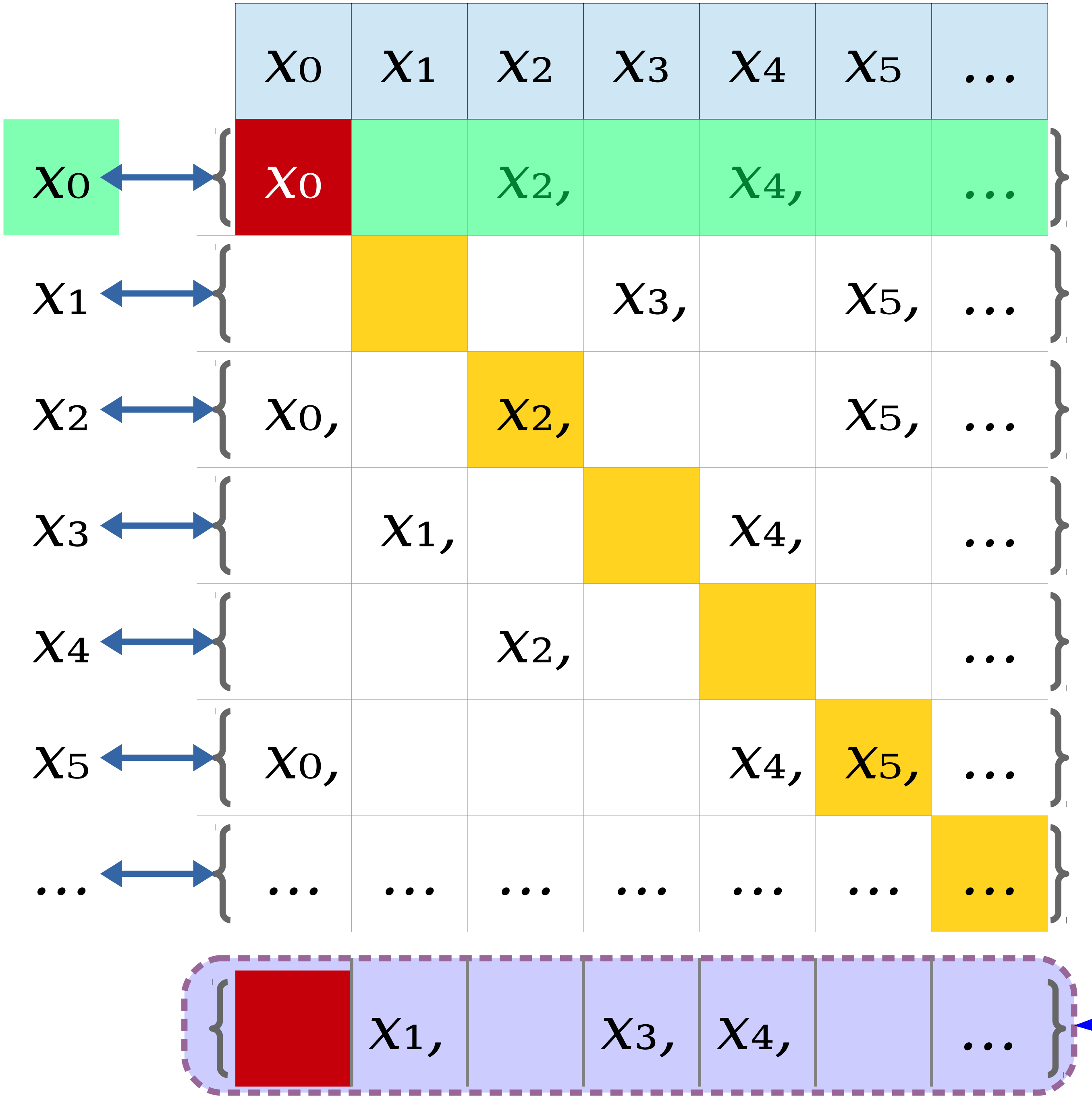


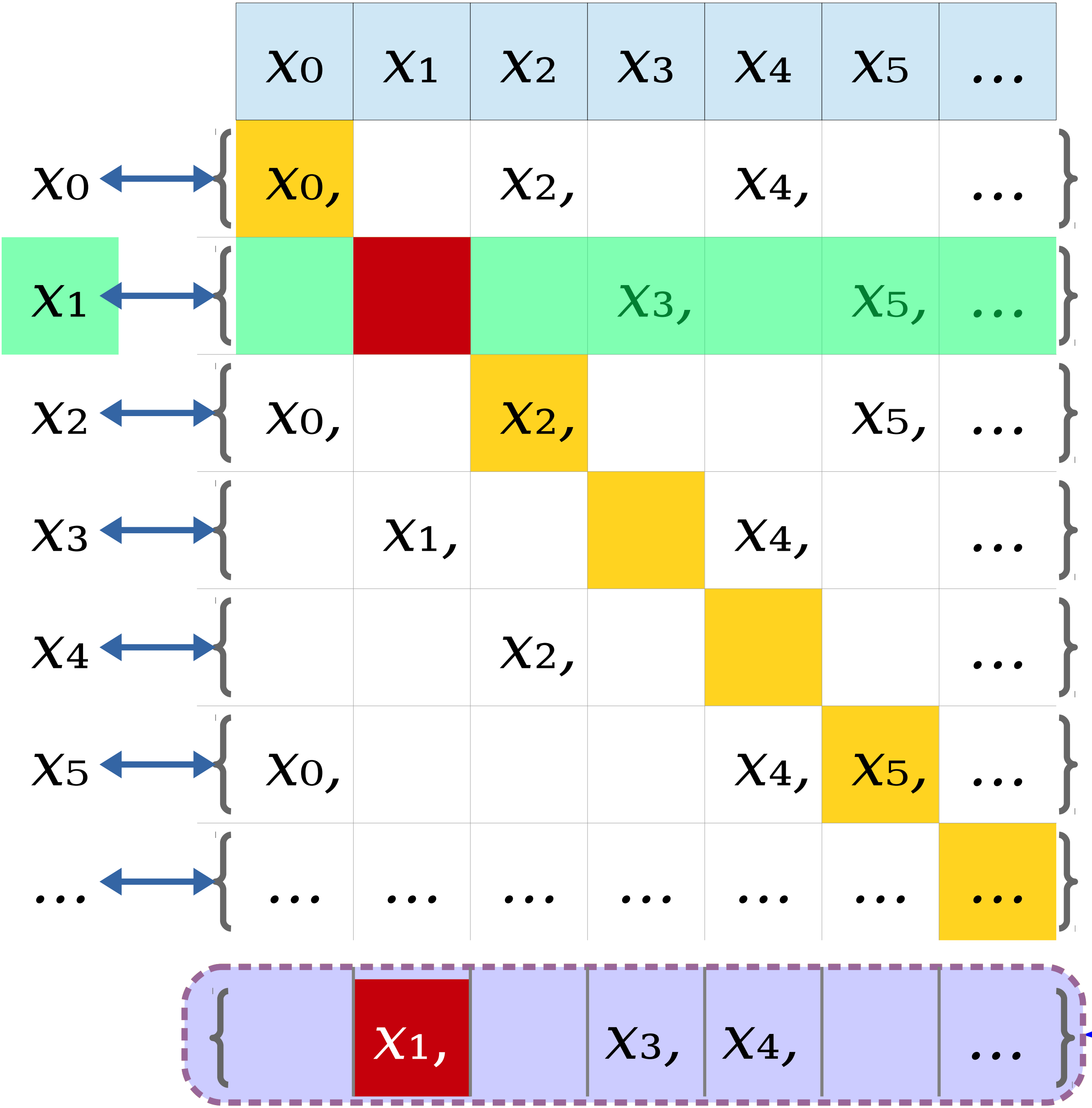
"Flip" this set. Swap what's included and what's excluded.

This is a drawing of our function $f : S \to \wp(S)$.



Which element is paired with this set?

This is a drawing of our function $f : S \to \wp(S)$.



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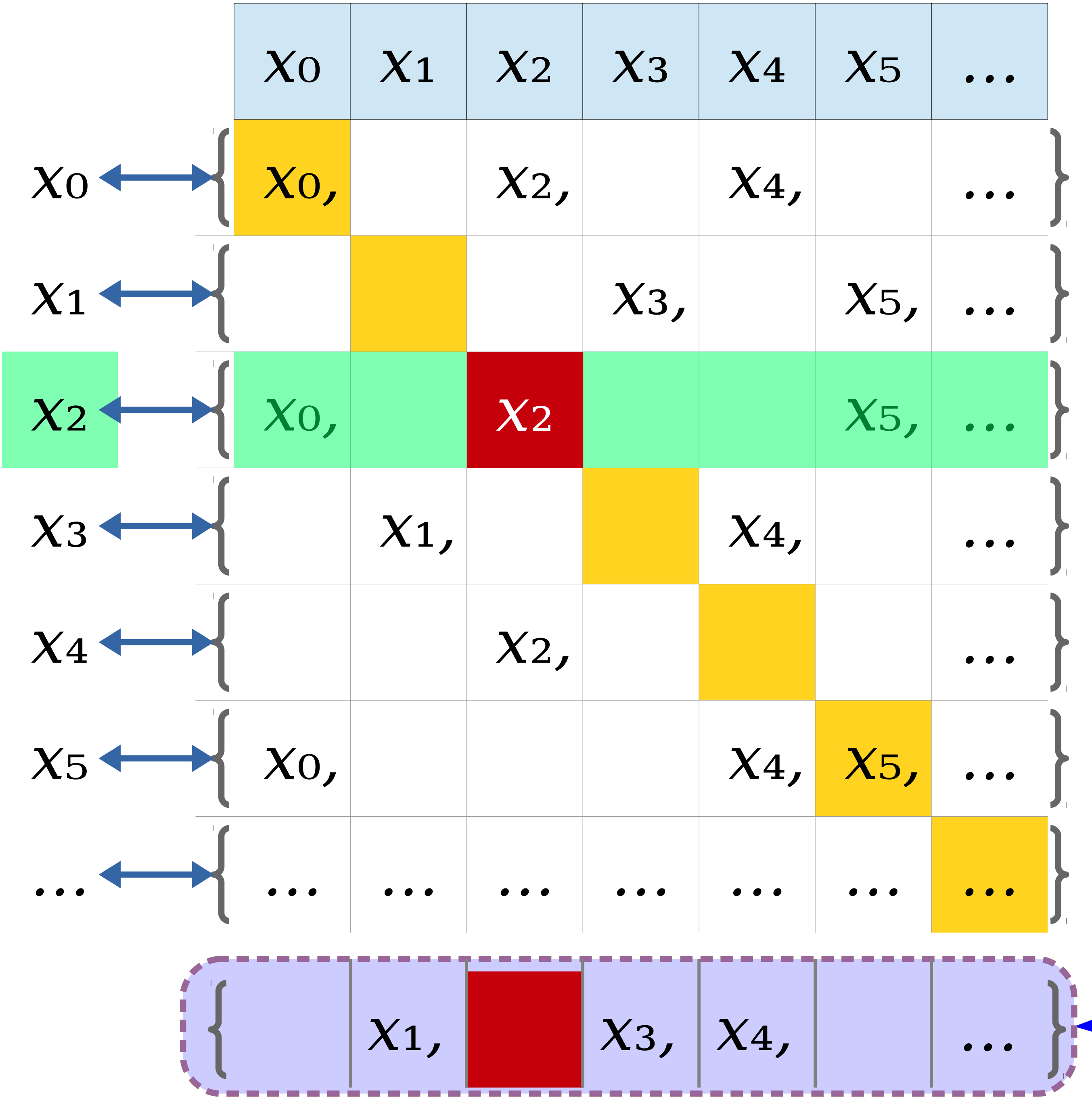

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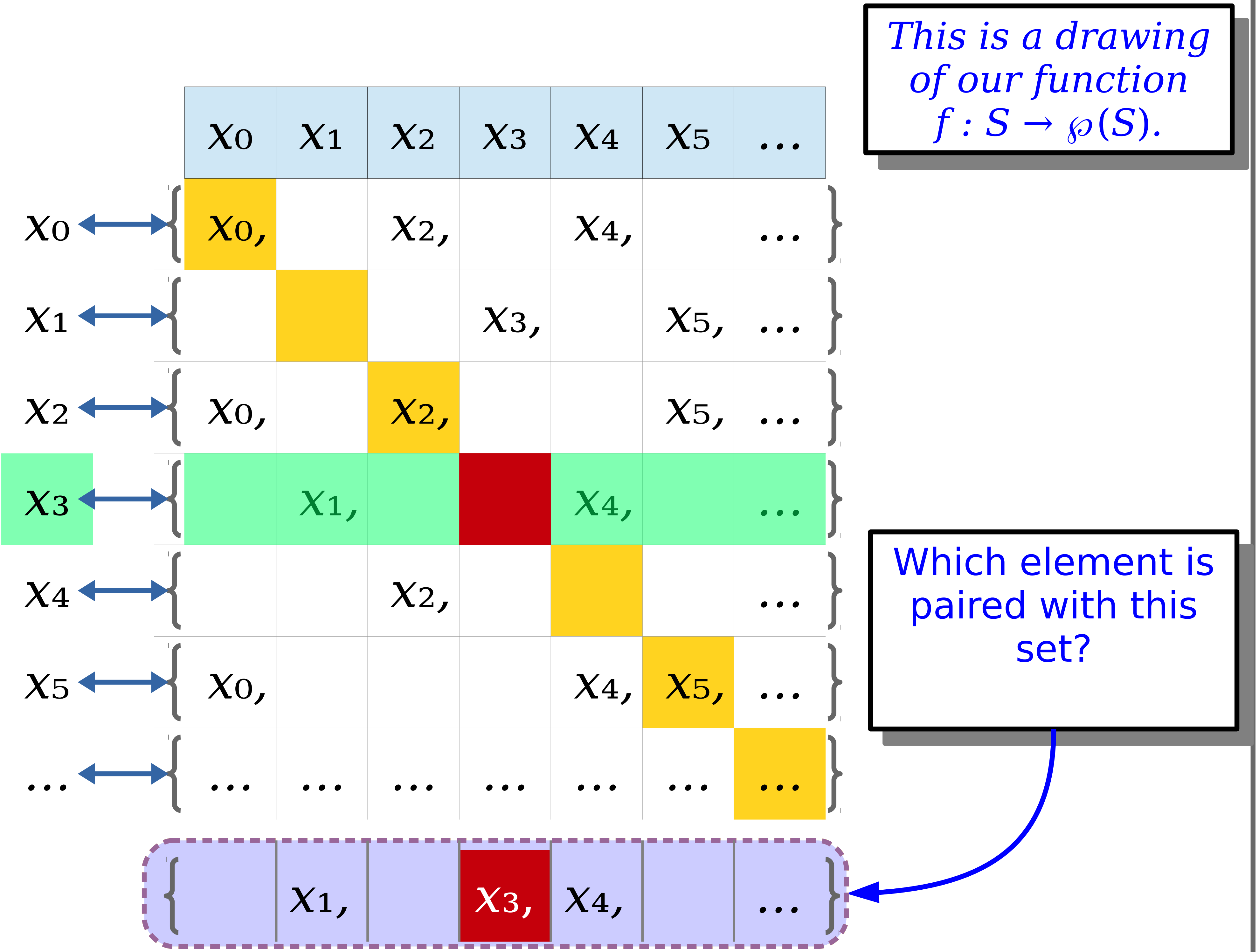

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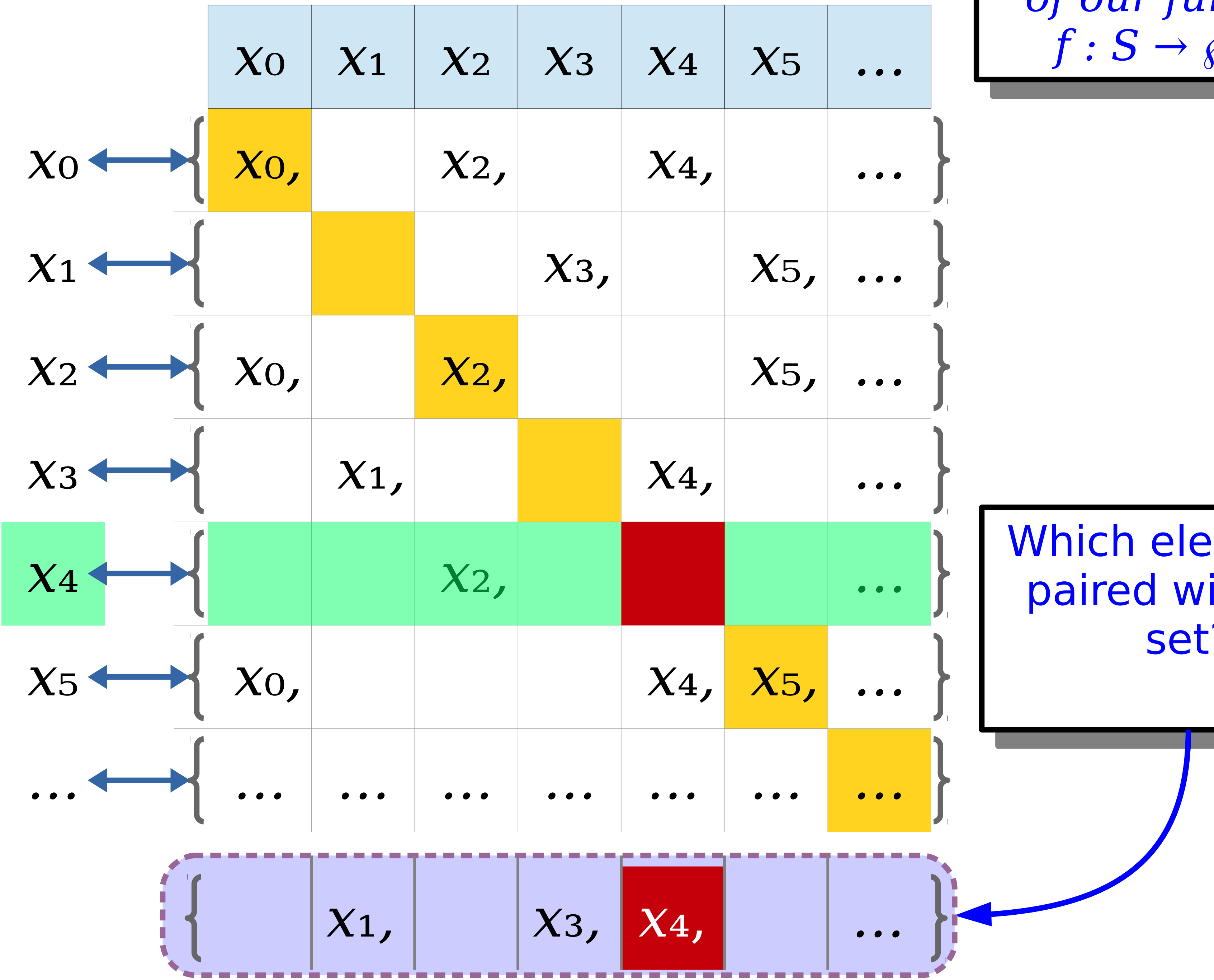

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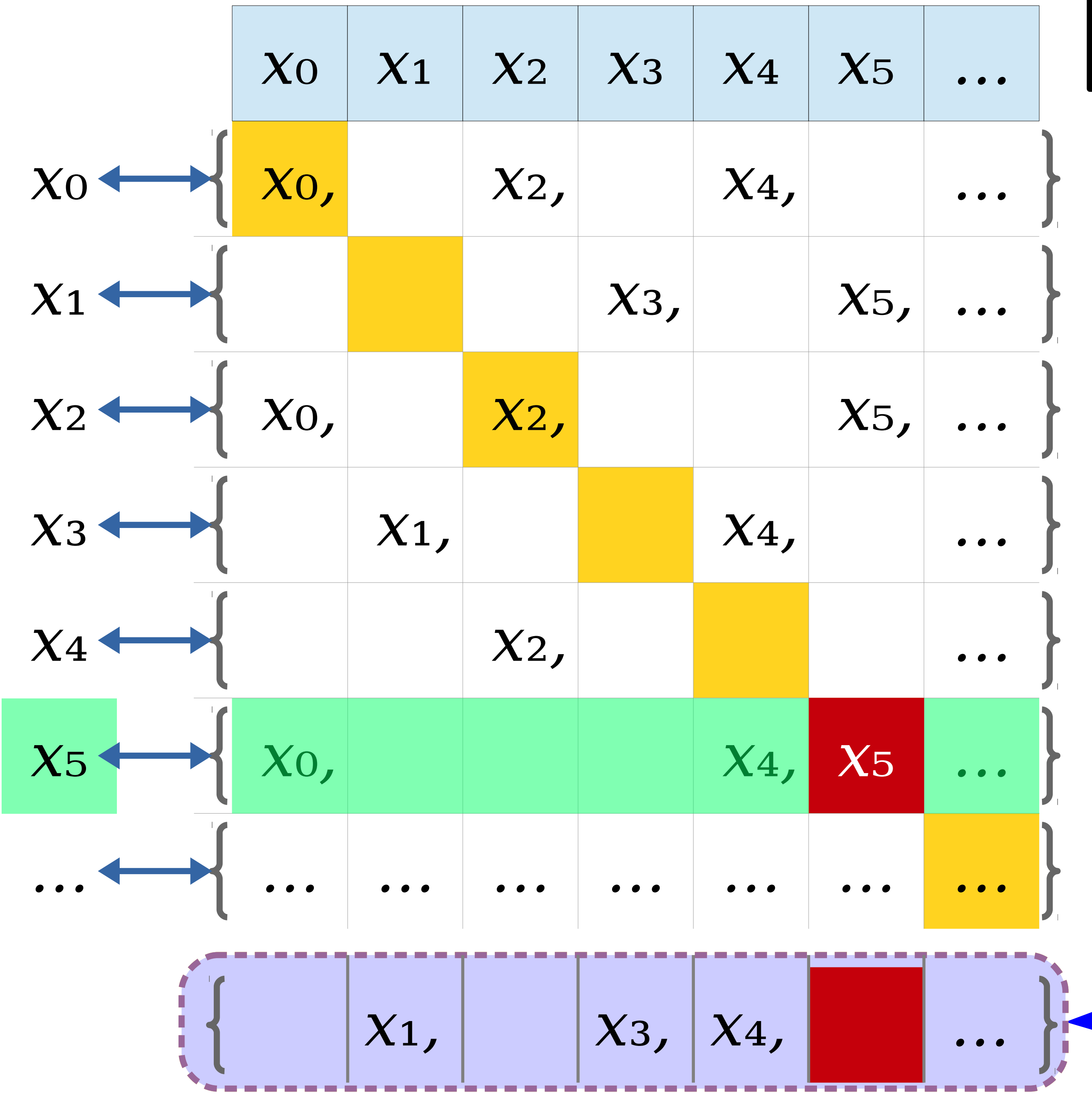

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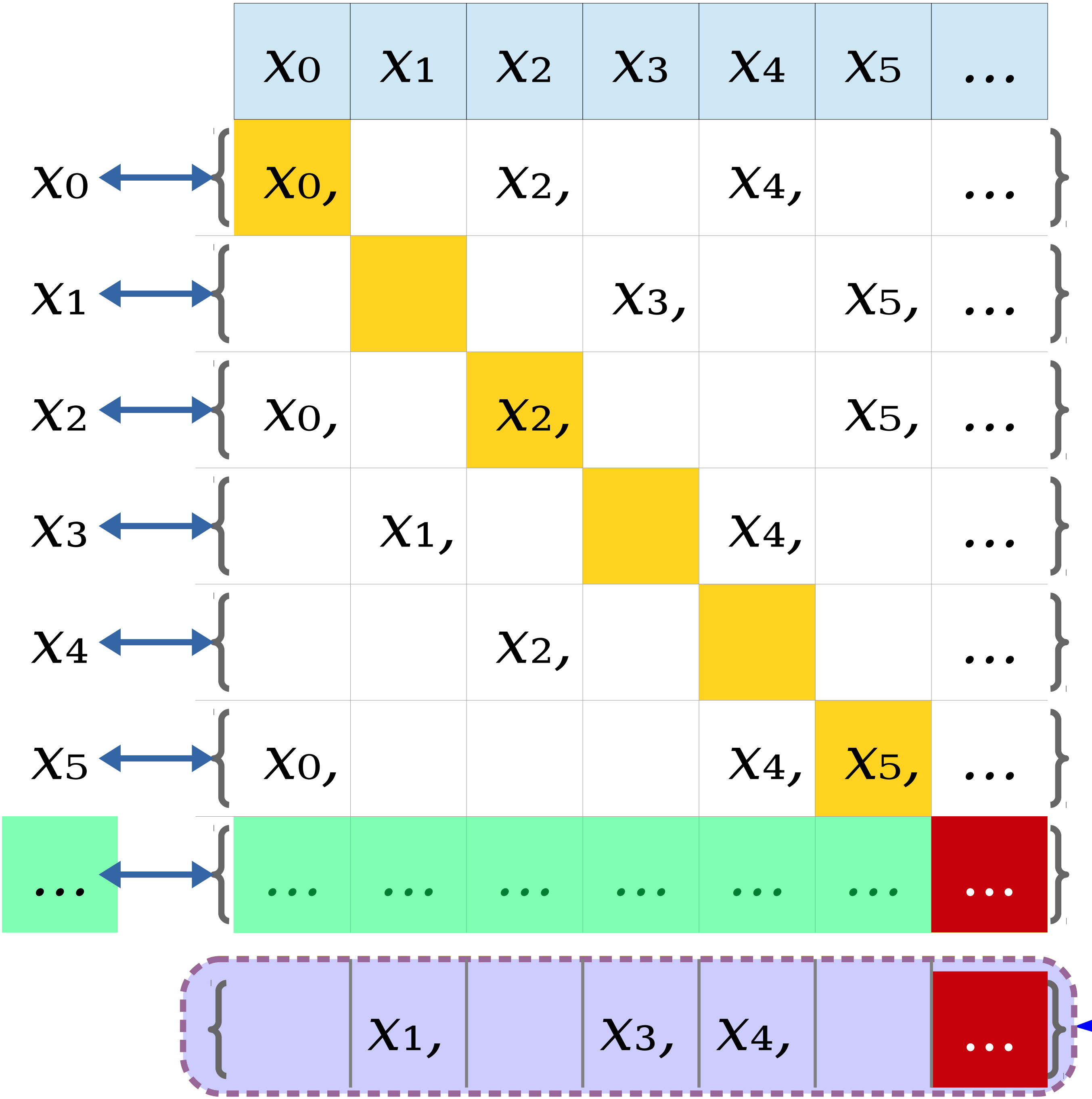
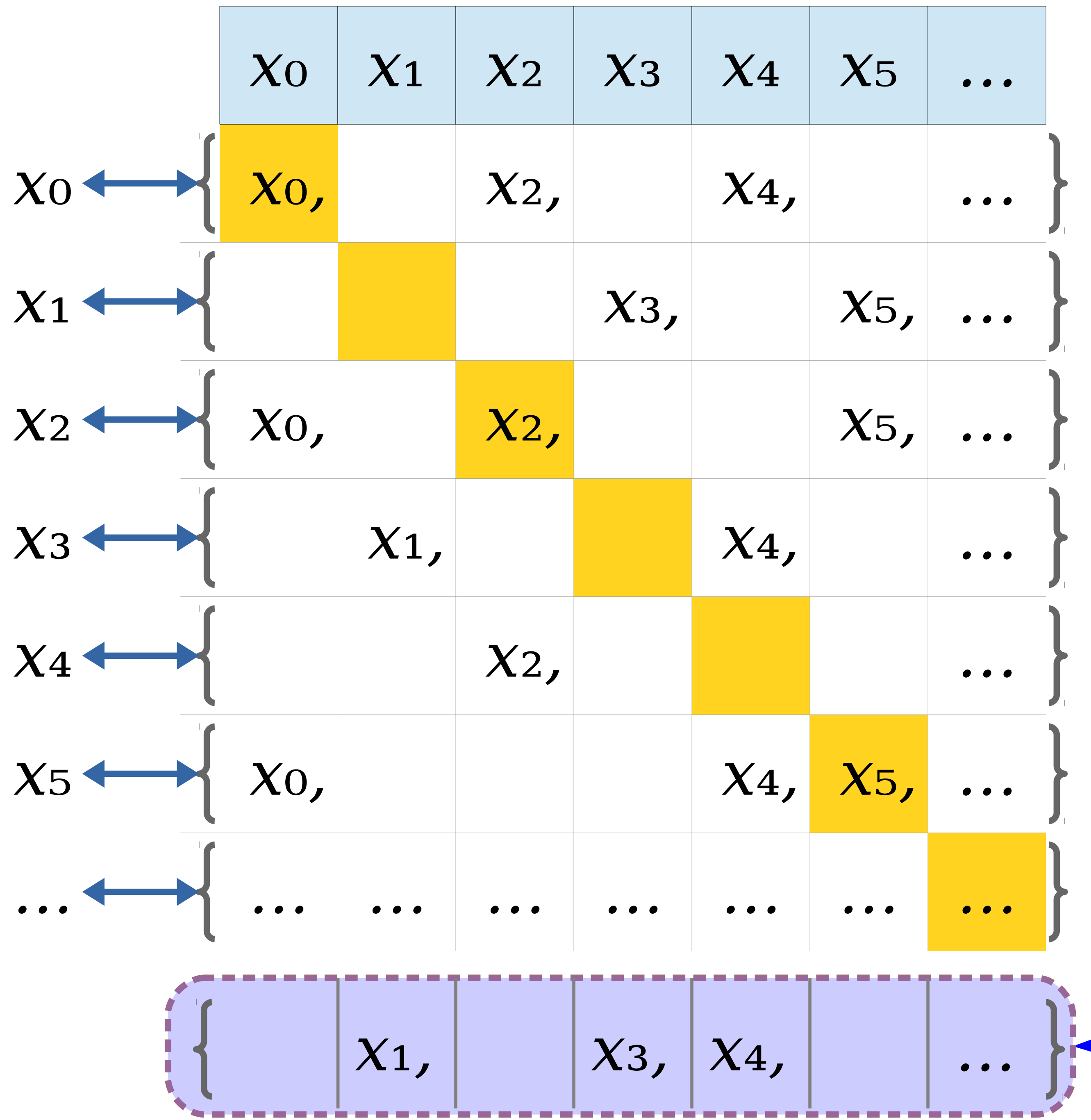


Which element is paired with this set?



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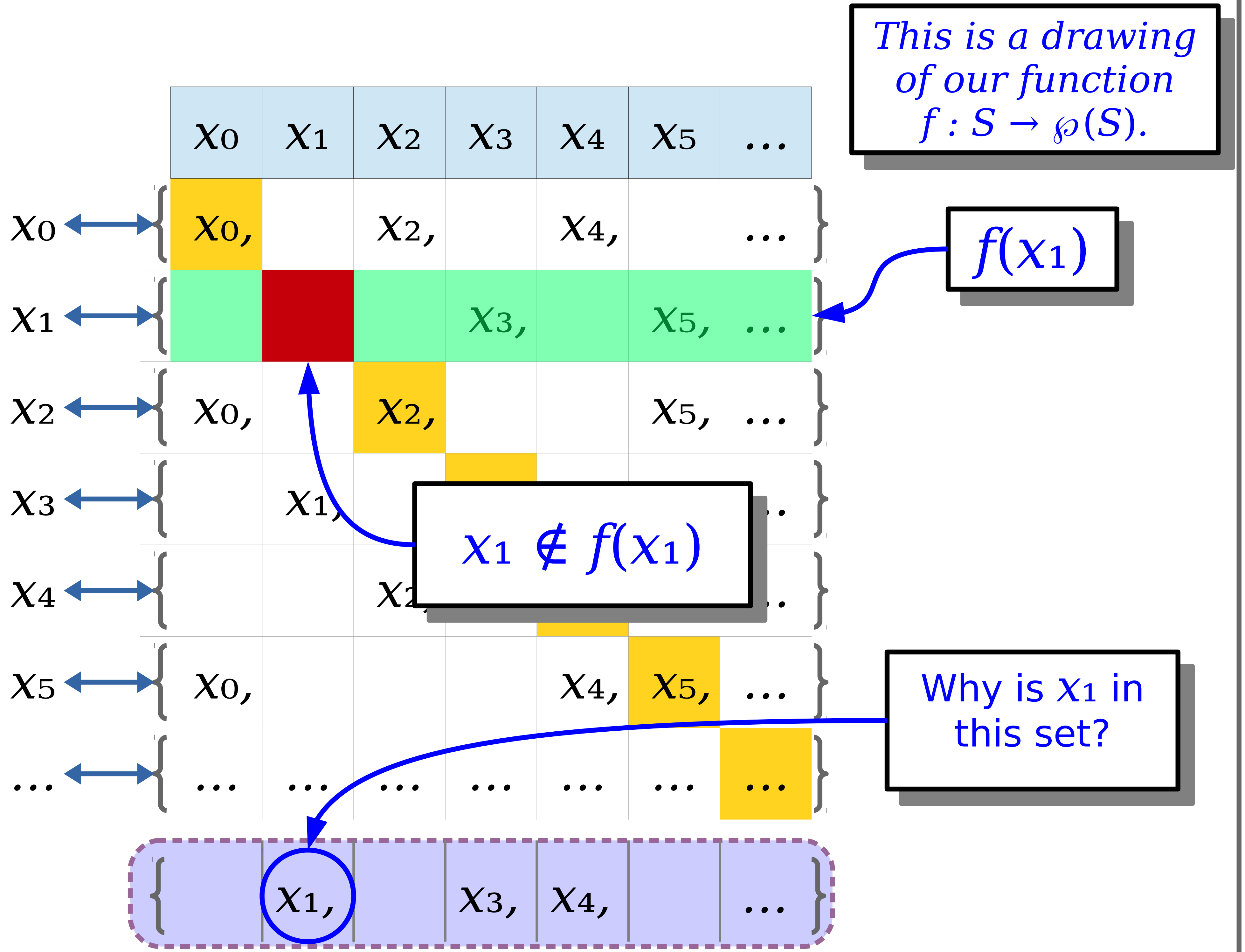
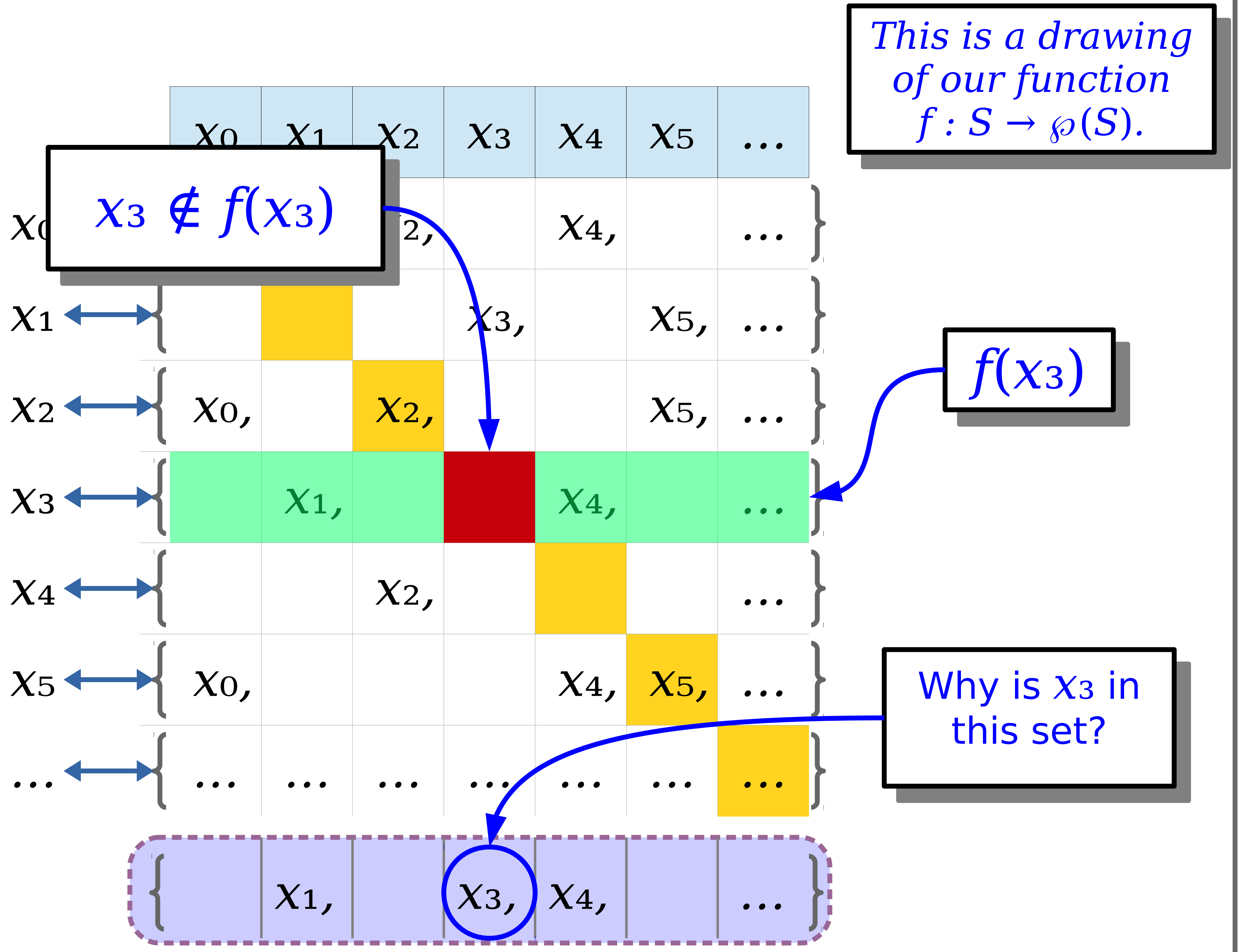
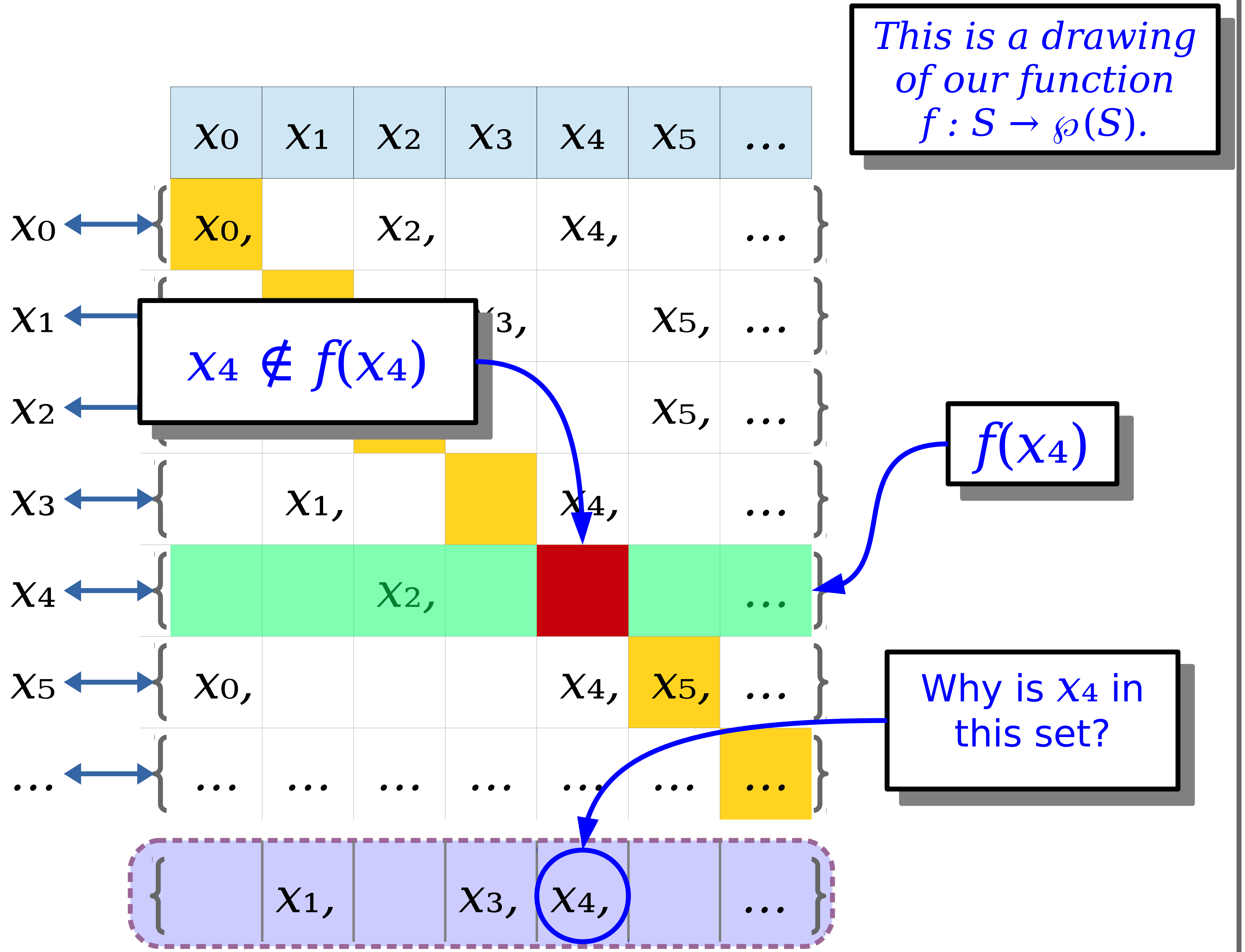
What set is this?



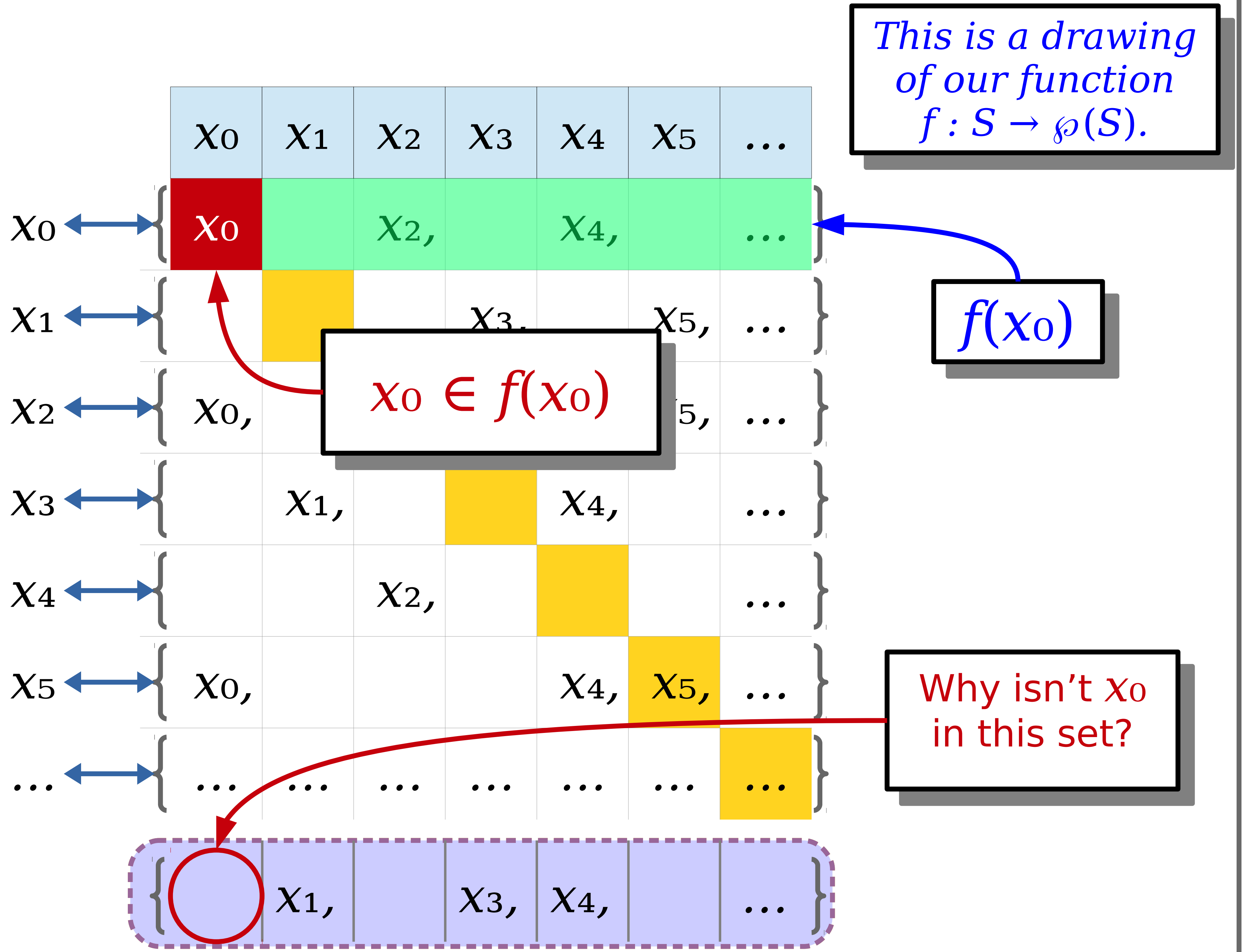
This is a drawing of our function $f : S \to \wp(S)$.
$f(x_1)$
$x_1 \notin f(x_1)$
Why is x_1 in this set?

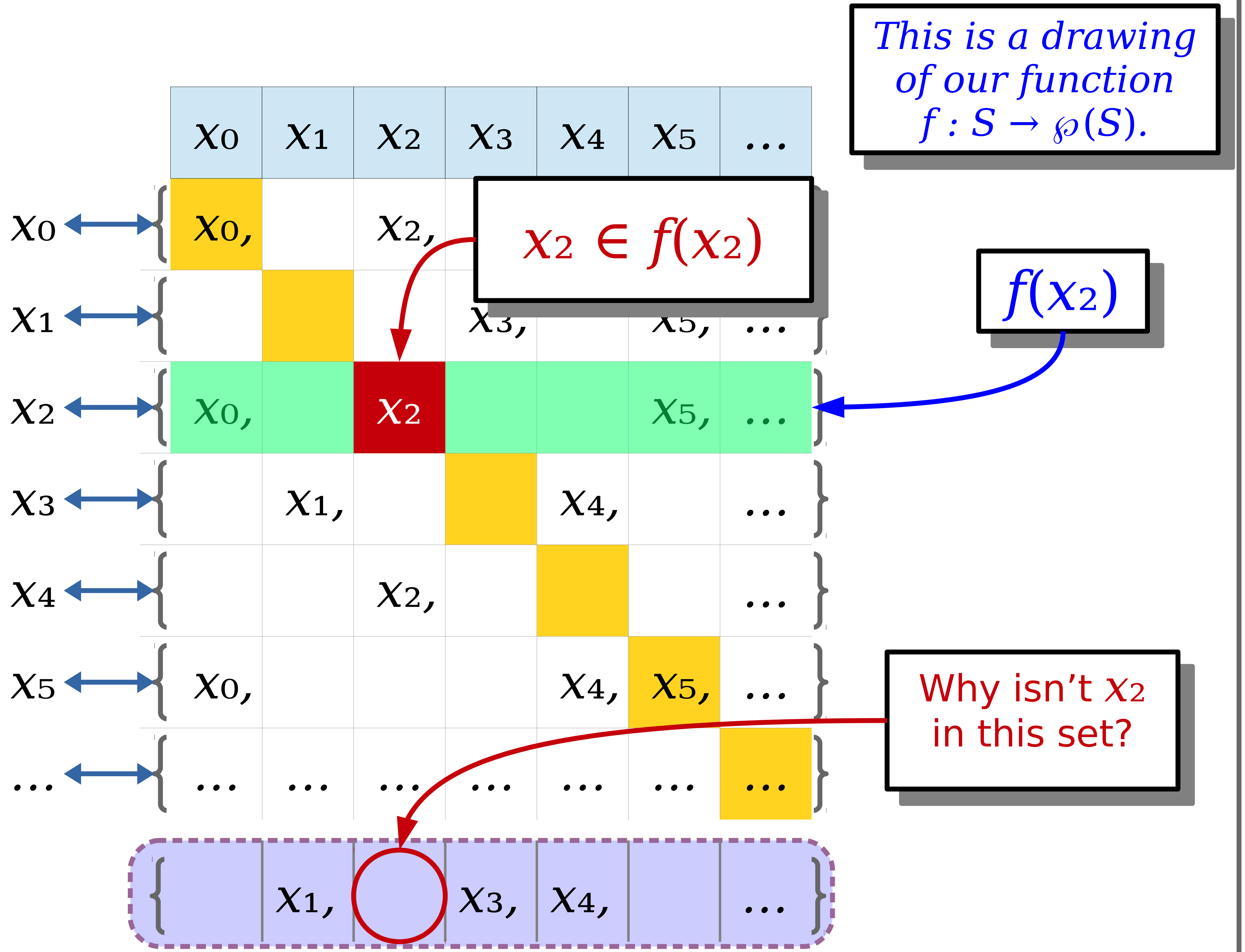


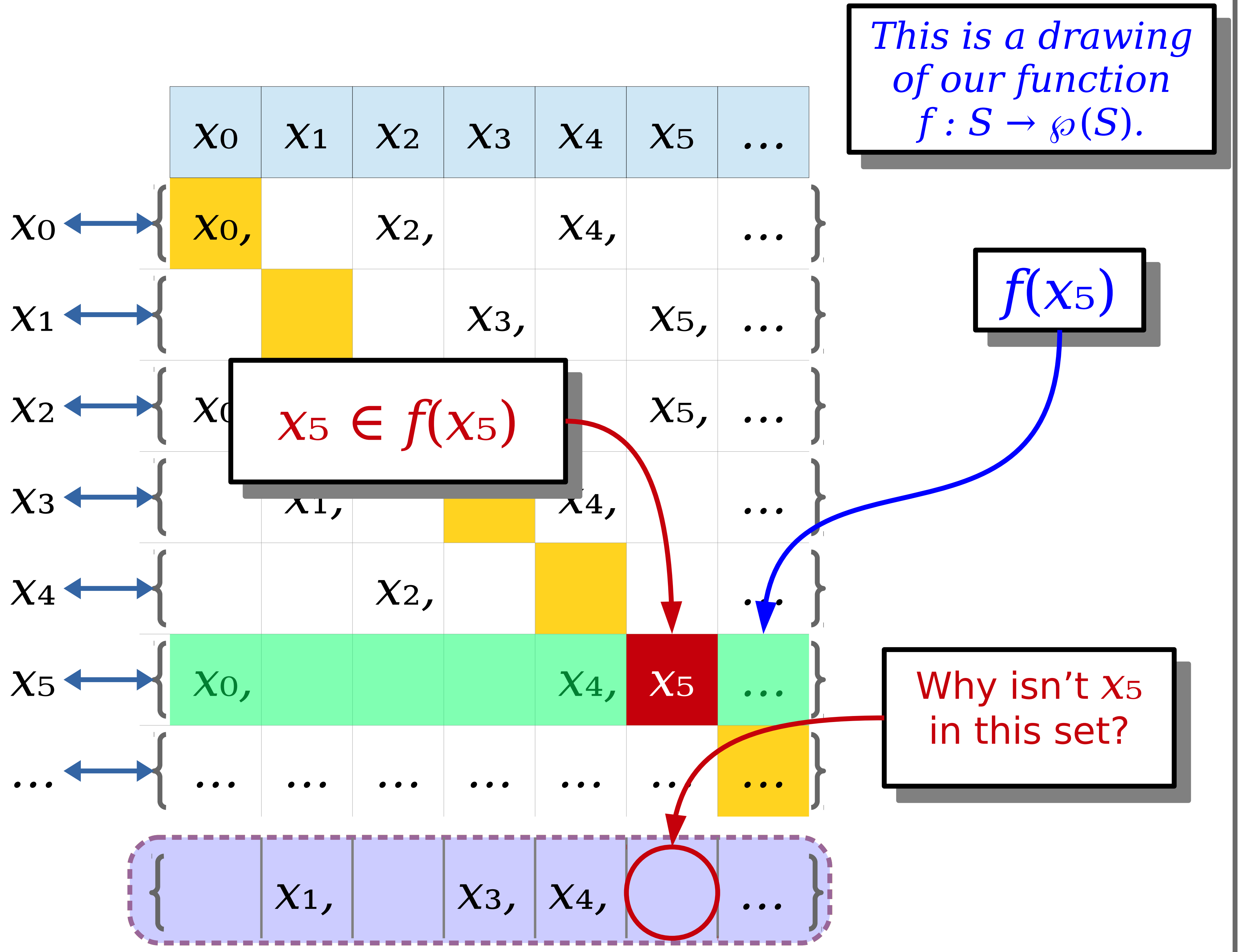
This is a drawing of our function $f : S \to \wp(S)$.
$x_3 \notin f(x_3)$
$f(x_3)$
Why is x_3 in this set?
x_0 x_1 x_2 x_3 x_4 x_5 …
$\{ x_1,\ x_3,\ x_4,\ \dots \}$



This is a drawing of our function $f : S \to \wp(S)$.
$x_4 \notin f(x_4)$
$f(x_4)$
Why is x_4 in this set?
x_0 x_1 x_2 x_3 x_4 x_5 ...
x_0 { x_0, x_2, x_4, ... }
x_1 { x_3, x_5, ... }
x_2 { x_5, ... }
x_3 { x_1, x_4, ... }
x_4 { x_2, ... }
x_5 { x_0, x_4, x_5, ... }
... { ... ... }
{ x_1, x_3, x_4, ... }

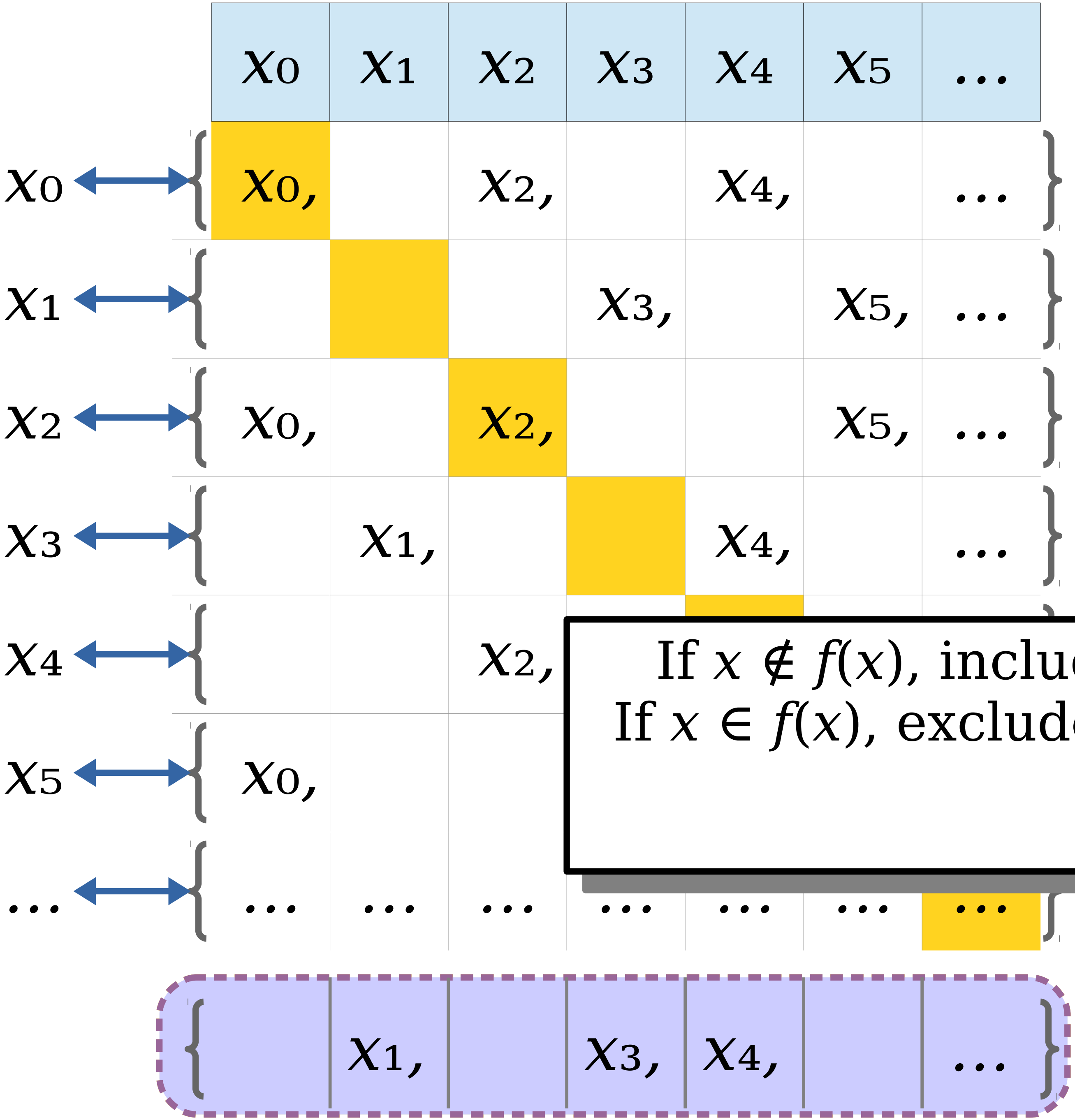

This is a drawing of our function $f : S \to \wp(S)$.
$f(x_0)$
$x_0 \in f(x_0)$
Why isn't x_0 in this set?
x_0 x_1 x_2 x_3 x_4 x_5 …
x_0 ↔ { x_0 x_2, x_4, … }
x_1 ↔ { x_3, x_5, … }
x_2 ↔ { x_0, x_5, … }
x_3 ↔ { x_1, x_4, … }
x_4 ↔ { x_2, … }
x_5 ↔ { x_0, x_4, x_5, … }
… ↔ { … … … … … … … }
{ x_1, x_3, x_4, … }


This is a drawing of our function $f : S \to \wp(S)$.
x_0 x_1 x_2 x_3 x_4 x_5 ...
$x_2 \in f(x_2)$
$f(x_2)$
Why isn't x_2 in this set?
{ x_1, x_3, x_4, ... }



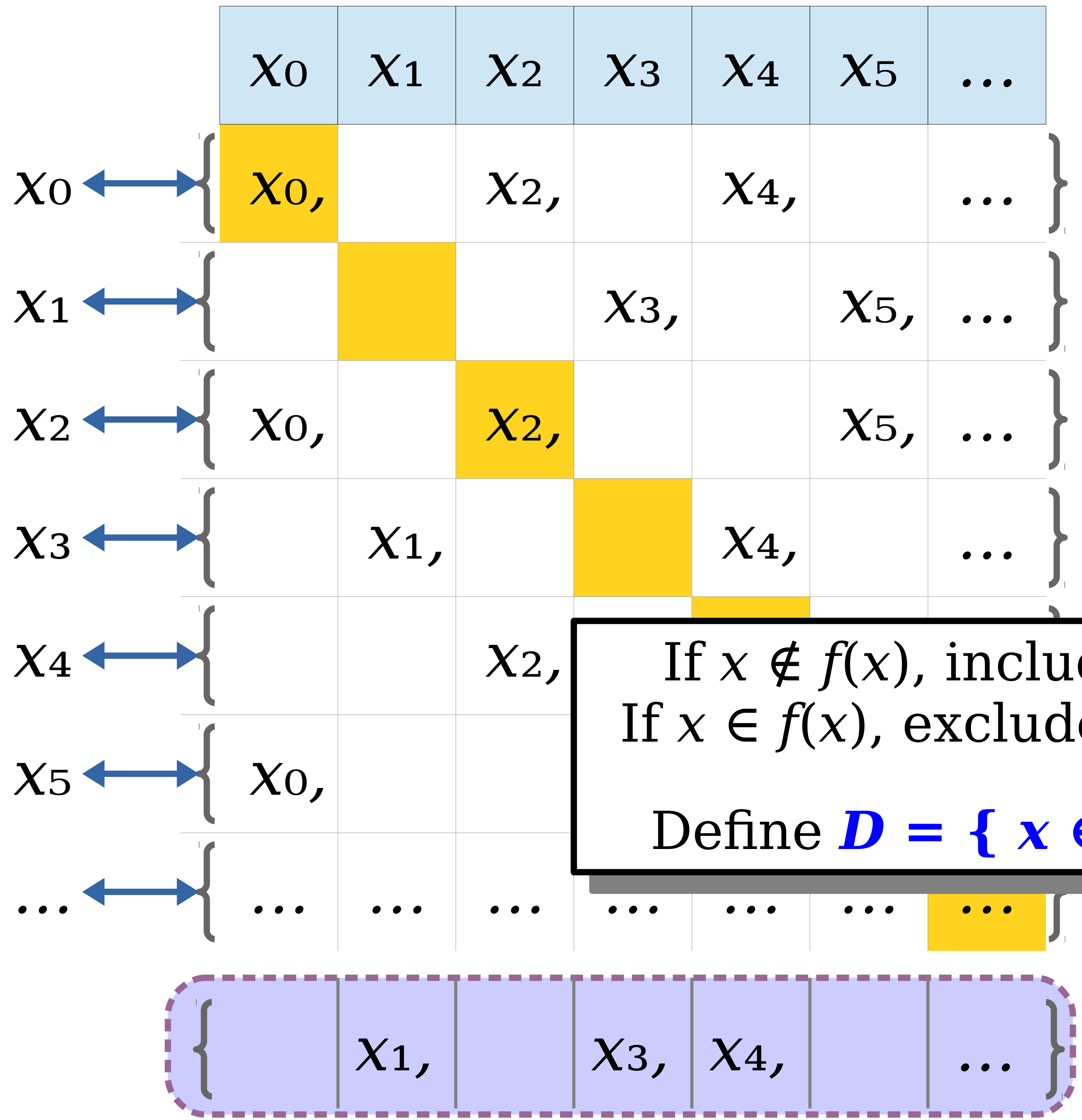
This is a drawing of our function $f : S \to \wp(S)$.
x_0 x_1 x_2 x_3 x_4 x_5 …
x_0 { x_0, x_2, x_4, … }
x_1 { x_3, x_5, … }
x_2 { x_5, … }
x_3 { x_4, … }
x_4 { x_2, … }
x_5 { x_0, x_4, x_5 … }
… { … … … … … … … }
{ x_1, x_3, x_4, … }
$f(x_5)$
$x_5 \in f(x_5)$
Why isn't x_5 in this set?

This is a drawing of our function $f : S \to \wp(S)$.



If $x \notin f(x)$, include x in the set.
If $x \in f(x)$, exclude x from the set.

This is a drawing of our function $f : S \to \wp(S)$.



If $x \notin f(x)$, include x in the set.
If $x \in f(x)$, exclude x from the set.

Define $\boldsymbol{D = \{ x \in S \mid x \notin f(x) \}}$

The Diagonal Set

- For any set S and function $f : S \to \wp(S)$, we can define a set D as follows:

$$\boldsymbol{D = \{\ x \in S \mid x \notin f(x)\ \}}$$

("The set of all elements x where x is not an element of the set f(x).")

- This is a formalization of the set we found in the previous picture.
- Using this choice of D, we can formally prove that no function $f : S \to \wp(S)$ is a bijection.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

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Starting with f, we define the set

$$D = \{ \ x \in S \mid x \notin f(x) \ \}. \qquad (1)$$

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We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \to \wp(S)$. We will prove that f is not surjective.

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We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

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Starting with f, we define the set

$$D = \{\ x \in S \mid x \notin f(x)\ \}. \qquad (1)$$

We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. By the definition of D, we know that

$$y \in D \text{ if and only if } y \notin f(y). \qquad (2)$$

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By assumption, $f(y) = D$.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

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The Big Recap

- We define equal cardinality in terms of bijections between sets.
- Lots of different sets of infinite size have the same cardinality.
- Cardinality acts like an equivalence relation – but only because we can prove specific properties of how it behaves by relying on properties of function.
- Cantor’s theorem can be formalized in terms of surjectivity.

Next Time

- ***Graphs***
  - A ubiquitous, expressive, and flexible abstraction!
- ***Properties of Graphs***
  - Building high-level structures out of lower-level ones!